1. Show that $L \subseteq \mathcal{A}^*$ is $\mathcal{NP}$-hard [respectively, $\mathcal{NP}^c$-complete] if and only if $(\mathcal{A}^* \setminus L)$ is $\mathcal{NP}^c$-hard [\(\mathcal{NP}^c\)-complete].

2. [For those of you familiar with the Propositional Calculus.] Show that the problem of deciding if a sentence of the Propositional Calculus is satisfiable is $\mathcal{NP}$-complete and the problem of deciding if a sentence is a tautology is $\mathcal{NP}^c$-complete.

3. Show that the language
$$\{ \sigma @ \tau @^k \in \{0, 1, @\}^* \mid \sigma, \tau \in \{0, 1\}^* \text{and } \sigma \text{ codes a NDTM } M_{\sigma} \text{ which has an accepting computation on input } \tau \text{ of less than or equal to } k \text{ steps} \}$$
is $\mathcal{NP}$-complete.

4. Show that $2SAT \in \mathcal{P}$. [Harder than your average question.]

5. Find a language which is both $\mathcal{NP}$-hard and $\mathcal{NP}^c$-hard but not recursive.

6. Suppose $f, g : \mathbb{N} \to \mathbb{N}$ with $g$ polynomial time computable, and there exists $n_0 \in \mathbb{N}$ such that $f(n) = g(n)$ for all $n > n_0$. Show that $f$ is polynomial time computable.

7. Design a suitable coding system for finite graphs. [Read the solution to this one, even if you are sure you have a correct answer!]

8. Show that the following problem is $\mathcal{NP}$-complete:

Given a graph $G$ and a natural number $k \in \mathbb{N}$, decide if $G$ has a set $C$ of $k$ vertices such that every pair of distinct vertices in $C$ are joined by an edge.

[A set $C$ of vertices with the given property is called a clique, and this problem is known as $\text{CLIQUE}$. Hint: given a Boolean expression $\bigwedge_{i=1}^m \bigvee_{j=1}^{s_i} y_{ij}$, consider the graph with $\{ \langle j, i \rangle \mid i \leq m, j \leq s_i \}$ and $\{ \langle j, i \rangle, \langle j', i' \rangle \} \in E \iff i \neq i'$ and $(y_{ij} \land y_{i'j'})$ satisfiable.]
9. Let NODECOVER be the problem:

Given a graph $G$ and a natural number $r \in \mathbb{N}$, decide if there is a set $N$ of $r$ vertices in $G$ such that every edge of $G$ has at least one end in $N$.

Show that NODECOVER is $\mathcal{NP}$-complete by showing that

$$\text{CLIQUE} \leq_{p} \text{NODECOVER}.$$  

10. Show that a language $L \subseteq A^{*}$ is in $\mathcal{NP}$ if and only if there exists an alphabet $B$, a function $f : L \rightarrow B^{*}$ with polynomial length expansion (i.e. $|f(\sigma)| \leq p(|\sigma|)$ for some polynomial $p(x)$), and a TM $N$ which runs in polynomial time and

- accepts all words $\sigma@f(\sigma)$ with $\sigma \in L$; and
- rejects all words $\sigma@\tau$ with $\sigma \notin L$.

Here, of course, @ is a new symbol assumed not to be in $A$ or $B$.

[Read the solution for some more remarks on why this fact is important.]

11. Show that if $X \in \mathcal{P}$ then $\mathcal{P}^{X} = \mathcal{P}$.

12. Show that if $X$ is $\mathcal{NP}$-complete then $\mathcal{NP}^{c} \subseteq \mathcal{P}^{X}$.

13. Let $M$ be an oracle TM and $f : \mathbb{N} \rightarrow \mathbb{N}$ a time constructible function. Show that there is a (multitape) oracle TM $N$ such that $N^{Y}$ runs in time $f(n)$ for all oracles $Y$, and $N^{Y}$ accepts a word $\sigma$ exactly if $M^{Y}$ accepts $\sigma$ in $< f(|\sigma|)$ steps.

14. Describe an oracle TM $M$ and three languages $X$, $Y$ and $Z$ such that $M^{X}$ runs in quadratic time, $M^{Y}$ always halts but not in polynomial time, and $M^{Z}$ never halts.

15. Let $K$ be a language. Find a language which is not accepted by $M^{K}$ for any oracle TM $M$ with oracle $K$.

16. Show that every $\text{PSpace}$-hard language is $\mathcal{NP}$-hard, and that the converse would imply $\mathcal{NP} = \text{PSpace}$. [Here, of course, a language $X$ is $\text{PSpace}$-hard if $L \leq_{p} X$ for every $L$ in $\text{PSpace}$.]

17. Show that $\text{PSPACE} = \text{NPSPACE} = \text{PSPACE}^{c}$. 