7 Method of steepest descent

This technique first developed by Riemann (1892)[1], is extremely useful for handling integrals of the form

$$I(\lambda) = \int_{\mathcal{C}} e^{\lambda p(z)} q(z)\, dz$$

where $\mathcal{C}$ is a contour in the complex plane and $p(z), q(z)$ are analytic functions, and λ is taken to be real.(If λ is complex ie $\lambda = |\lambda| e^{i\alpha}$ we can absorb the exponential factor into $p(z)$.) We require the behaviour of $I(\lambda)$ as $\lambda \to \infty$.

The basic idea of the method of steepest descent (or sometimes referred to as the *saddle-point method*), is that we apply Cauchy's theorem to deform the contour $\mathcal{C}$ to contours coinciding with the path of steepest descent. Usually these contours pass through points $z = z_0$ where $p'(z_0) = 0$. As we will see on the steepest descent contours, $\Im(p(z))$ is constant and so we are left with integrals of the type which can be handled using Watson's lemma.

Let $p(z) = u(x, y) + iv(x, y)$ be an analytic function of the complex variable $z = x + iy$ in some domain $\mathcal{D}$. Notice that for any path of integration the exponential function

$$e^{\lambda p(z)} = e^{\lambda u(x,y)} e^{i\lambda v(x,y)}$$

may have a maximum modulus at some point $z = z_0$ on the path. Ideally we would like to choose a path near a point $z = z_0$ such that u attains a peak and decreases away from $z = z_0$. But the imaginary part $v(x, y)$ will in general also change and the exponential factor $e^{i\lambda v}$ will oscillate rapidly near z_0.

This suggests that a suitable path is one where $v(x, y)$ is nearly constant as we move away from $z = z_0$. Also by the maximum modulus theorem u, v cannot attain maximum or minimum values in a domain if $p(z)$ is analytic, only on the boundary of the region. Thus the point $z = z_0$ must coincide with a saddle point where $p'(z_0) = 0$.

The method of steepest descent is thus also called the saddle point method. If we consider the surface

$$u(x, y) = u(x_0, y_0)$$

passing through some point $z = z_0$ of $\mathcal{D}$ then note that $\nabla u|_{z_0}$ defines the direction of steepest ascent from the point $z = z_0$ and $-\nabla u|_{z_0}$ the direction of steepest descent.

Now consider the surface

$$v(x, y) = v(x_0, y_0).$$

We have that $\nabla v = (v_x, v_y)$ is in a direction normal to the surface. But using the Cauchy-Riemann equations $v_x = -u_y, v_y = u_x$.

[1] B.Riemann, *Gesammelte Mathematische Werke* 2e Aufl, Leipzig, pp. 424-430

Thus

$$\nabla v = (v_x, v_y) = (-u_y, u_x).$$

Hence a direction tangential to the surface is given by

$$(-v_y, v_x) = (u_x, u_y) = \nabla u.$$

Thus tangents to the surface

$$v(x, y) = v(x_0, y_0)$$

lie in the direction of steepest ascent/descent through z_0 and the lines of constant u and constant v intersect at right angles in the regions of analyticity of the functions.

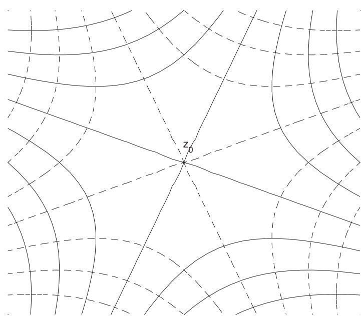


Figure 2: Typical steepest ascent/descent curves shown by solid/dashed lines near a simple saddle point $z = z_0$

Observe also that for any change δp

$$\delta p = \delta u + i\delta v$$

and so

$$|\delta u| \leq |\delta p|$$

and $|\delta u|$ is a maximum at $z = z_0$ only when $\delta v = 0$, ie when $v(x, y) = v(x_0, y_0)$.

Definition *We define $z = z_0$ to be* **a saddle point of order** $N - 1$ *if*

$$p'(z_0), \ldots p^{(N-1)}(z_0) = 0, \quad p^{(N)}(z_0) \neq 0.$$

A saddle point or order 1 is a simple saddle point.

Near $z = z_0$ we have

$$p(z) = p(z_0) + \frac{(z-z_0)^N}{N!} p^{(N)}(z_0) + o((z-z_0)^N).$$

Putting

$$z = z_0 + \rho e^{i\theta}, \quad p^{(N)}(z_0) = a e^{i\alpha}$$

we have

$$p(z) - p(z_0) \sim \frac{\rho^N a e^{i(N\theta+\alpha)}}{N!}.$$

Thus the curves of steepest ascent/descent through $z = z_0$ are given locally by

$$\Im(p(z) - p(z_0)) = 0 \qquad \Longrightarrow \sin(N\theta + \alpha) = 0,$$

giving $N\theta + \alpha = k\pi$, where k is an integer. In this case

$$p(z) - p(z_0) = u(x,y) - u(x_0,y_0) \sim \frac{\rho^N}{N!} a \cos(N\theta + \alpha).$$

Thus the curves of **steepest descent** are given by

$$\theta = -\frac{\alpha}{N} + (2k+1)\frac{\pi}{N} \quad k = 0, 1, 2, \ldots, N-1,$$

since $\cos(N\theta + \alpha)$ is then negative and $u(x,y) < u(x_0,y_0)$ as we move away from $z = z_0$.

The curves of **steepest ascent** are given by

$$\theta = -\frac{\alpha}{N} + 2k\frac{\pi}{N} \quad k = 0, 1, 2, \ldots, N-1,$$

since $\cos(N\theta + \alpha)$ is then positive and $u(x,y) > u(x_0,y_0)$ as we move away from $z = z_0$.

Example Consider $p(z) = z - \frac{z^3}{3}$. Now $p'(z) = 1 - z^2$ and so the *critical points* where $p'(z) = 0$ are given by $z = \pm 1$. Also $p''(z) = -2z$ and thus $p''(1) = -2 = 2e^{i\pi}$, and $p''(-1) = 2$.

Near $z = 1$ the directions of steepest descent are given by the directions $\theta = -\frac{\pi}{2} + (2k+1)\frac{\pi}{2}, \quad k = 0, 1$ ie $\theta = 0, \pi$.

Near $z = -1$ the directions of steepest descent are given by the directions $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Consider the point $z = 1$. The steepest descent/ascent curves satisfy

$$v(x,y) = \Im(z - \frac{z^3}{3}) = y(1 - x^2 + \frac{y^2}{3}) = v(1,0) = 0.$$

There are two curves of steepest ascent/descent passing through $z_0 = 1$. These are $y = 0$ and $1 - x^2 + \frac{y^2}{3} = 0$. Clearly $y = 0$ is the steepest descent curve, see Fig. 7.

Next consider the point $z = -1$. Here the steepest descent/ascent curves satisfy

$$v(x, y) = y(1 - x^2 + \frac{y^2}{3}) = v(-1, 0) = 0.$$

This time the curve $1 - x^2 + \frac{y^2}{3} = 0$ is the curve of steepest descent emanating from $x = -1$.

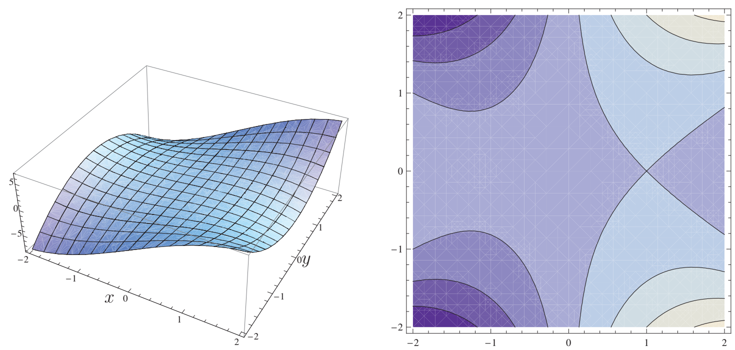


Figure 3: Plots of the surface and contour levels for $u(x, y) = Re(z - z^3/3) = u(1, 0)$. The steepest descent path is given by $y = 0$.

Example

Consider $p(z) = \cosh z - \frac{z^2}{2}$. Here

$$p'(z) = \sinh z - z, \quad p''(z) = \cosh z - 1, \quad p'''(z) = \sinh z.$$

Thus $z = 0$ is a saddle point of order 4 and

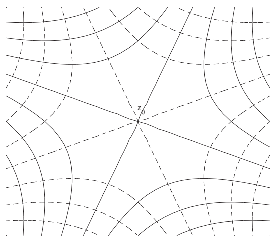
$$p''''(0) = 1,$$

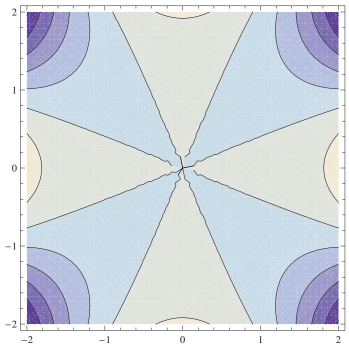
and the directions of steepest descent are $\theta = (2k+1)\frac{\pi}{4}, \quad k = 0, 1, 2, 3$. A plot of the surface $u(x, y) = \Re(p(z)) - 1$ is shown in Fig. 7.

7.1 Method of steepest descent- key steps

The key steps in using the method of steepest descent are:

- Identify the saddle points, singular points and endpoints likely to contribute to an estimate for the integral.
- Determine path of steepest descent. It may be the case that there is no continuous path joining the endpoints and one needs two or more steepest descent paths.





- Deform contour making use of Cauchy's theorem.
- Evaluate integral making use of Watson's lemma as appropriate.

The books by Bender & Orszag[1], (chapter 6), and Bleistein & Handelsmann[1] (chapter 7) contain many examples which should be studied in detail.

The first example below is taken from Bender and Orszag.

Example

Consider

$$I(\lambda) = \int_0^1 e^{i\lambda t} \log t \, dt.$$

Here

$$p(z) = iz = ix - y.$$

There are no saddle points. The steepest descent/ascent paths are given by

$$\Im(p(z)) = x = constant.$$

Since $u(x, y) = \Re(p(z)) = -y$, for $y > 0$ the curves $x = constant$ are steepest descent paths.

Also note that there is no continuous steepest descent path passing through the two endpoints of the integral $x = 0$ and $x = 1$.

This motivates the choice of the contour $C_1 + C_2 + C_3$ that we deform the original path of integration, see Fig. 4. We take a branch cut along the negative real axis for $\log(z)$.

Using Cauchy's theorem we can write

$$\int_0^1 \log(t) e^{i\lambda t} \, dt = -\int_{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3} e^{i\lambda z} \log z \, dz.$$

[1]see references in lecture 1

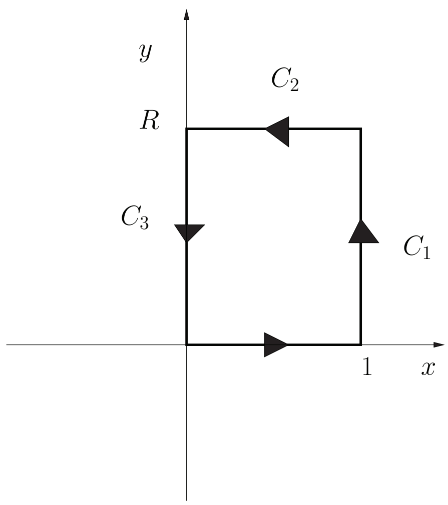


Figure 4: Deformed contour for integral $\int_0^1 \log t e^{i\lambda t}\,dt$.

For C_2 put $z = x + iR$ and then

$$\int_{\mathcal{C}_2} = -\int_0^1 \log(x+iR)e^{i\lambda(x+iR)}\,dx$$

and we see that

$$|\int_{\mathcal{C}_2}| \le e^{-R\lambda}\int_0^1 |\log(x+iR)|\,dx$$

and thus goes to zero as $R \to \infty$.

For $\mathcal{C}_1$ put $z = 1 + iy$ then

$$\int_{\mathcal{C}_1} = i\int_0^R \log(1+iy)e^{i\lambda(1+iy)}\,dy = ie^{i\lambda}\int_0^R \log(1+iy)e^{-y\lambda}\,dy.$$

For $\mathcal{C}_3$ put $z = iy$ then

$$\int_{\mathcal{C}_3} = -i\int_0^R \log(iy)e^{-\lambda y}\,dy.$$

Letting $R \to \infty$ we find that

$$I(\lambda) = i\int_0^\infty e^{-\lambda y}\log(iy)\,dy - ie^{i\lambda}\int_0^\infty \log(1+iy)e^{-\lambda y}\,dy.$$

Now

$$\begin{aligned} i\int_0^\infty e^{-\lambda y}\log(iy)\,dy &= i\int_0^\infty e^{-\lambda y}(\log y + \frac{i\pi}{2})\,dy \\ &= -\frac{\pi}{2\lambda} + i\frac{1}{\lambda}\int_0^\infty (\log(y) - \log(\lambda))e^{-y}\,dy, \end{aligned}$$

$$= -\frac{\pi}{2\lambda} - \frac{i\log\lambda}{\lambda} + \frac{i\gamma}{\lambda}$$

where γ is the Euler constant and we have used the result that

$$-\gamma = \int_0^\infty \log y e^{-y}\, dy.$$

Next applying Watson's lemma to the integral

$$I_1 = ie^{i\lambda}\int_0^\infty \log(1+iy)^{-\lambda y}\, dy$$

gives

$$I_1 \sim ie^{i\lambda}\int_0^\infty \sum_{n=1}^\infty (-1)^{n-1}\frac{(iy)^n}{n}e^{-\lambda y}\, dy,$$

$$I_1 \sim e^{i\lambda}\sum_{n=1}^\infty \frac{(-i)^{n+1}\Gamma(n+1)}{n\lambda^{n+1}}.$$

Hence putting it all together

$$I(\lambda) = \int_0^1 \log(t)e^{i\lambda t}\, dt \sim -\frac{i\log\lambda}{\lambda} - \frac{2i\gamma+\pi}{2\lambda} + e^{i\lambda}\sum_{n=1}^\infty \frac{(-1)^n(i)^{n+1}\Gamma(n)}{\lambda^{n+1}}.$$

Example

Consider the integral

$$I(\lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda(at-t^{\frac{1}{2}})}}{t}\, dt,$$

where a, c are positive constants and we take a branch cut along the negative real axis.

We require the behaviour of $I(\lambda)$ as $\lambda \to \infty$. Here

$$p(t) = at - t^{\frac{1}{2}}, \quad p'(t) = a - \frac{1}{2}t^{-\frac{1}{2}}, \quad p''(t) = \frac{1}{4}t^{-\frac{3}{2}}.$$

There is a simple saddle point given by $p'(t) = 0$ ie at $t = t_0 = 1/(4a^2)$.

Note that $p''(t_0) = 2a^3$ and so the steepest descent paths have directions $\theta = \pi/2, 3\pi/2$ emanating from the $t = t_0$.

By Cauchy's theorem we can deform the path of integration to pass through the saddle point as shown in the Fig. 5.

Thus to obtain the leading order estimate for the integral, we can approximate the SD path by a straight line in the direction of steepest descent, ie put $t = \frac{1}{4a^2} + iT$ and note that

$$at - t^{\frac{1}{2}} = -\frac{1}{4a} - \frac{2a^3}{2}T^2 + \ldots,$$

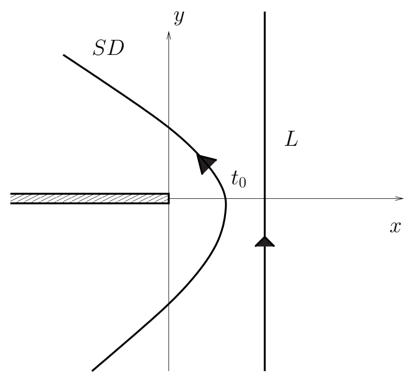


Figure 5: Original path L is deformed to the steepest descent path SD passing through the saddle point $t = t_0 = 1/(4a^2)$.

and

$$\frac{1}{t} = 4a^2 + \dots .$$

Thus the integral becomes

$$I(\lambda) = \frac{1}{2\pi i} \int_{SD} \frac{e^{\lambda(at - t^{\frac{1}{2}})}}{t} \, dt,$$

$$\sim \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{\lambda}{4a}} e^{-a^3 T^2 \lambda} 4a^2 i \, dT.$$

Hence

$$I(\lambda) \sim e^{\frac{-\lambda}{4a}} \frac{2a^2}{\pi} \int_{-\infty}^{\infty} e^{-a^3 T^2 \lambda} \, dT = 2\sqrt{\frac{a}{\pi\lambda}} e^{-\frac{\lambda}{4a}}.$$

To obtain more terms one needs to work harder.

First note that the steepest descent paths satisfy

$$p(t) = p(t_0) \quad \Longrightarrow \; at - t^{\frac{1}{2}} = -\frac{1}{4a},$$

and so the imaginary part of $p(t)$ is zero along these paths. Hence if we put

$$-W = at - t^{\frac{1}{2}} + \frac{1}{4a}$$

where W is real and positive, (because we have a SD path) we find that

$$at - t^{\frac{1}{2}} + \frac{1}{4a} + W = 0$$

giving

$$t = \left[\frac{1 \pm [1 - 4a(\frac{1}{4a} + W)]^{\frac{1}{2}}}{2a} \right]^2 = \left(\frac{1 \pm 2a^{\frac{1}{2}} W^{\frac{1}{2}}}{2a} \right)^2 .$$

The $\pm$ signs here indicate the two steepest directions emanating from $t = t_0$, see Fig. 6 and by expanding for small W we see that $+$ sign corresponds to the $\pi/2$ direction and the $-$ sign the $3\pi/2$ direction. We can write

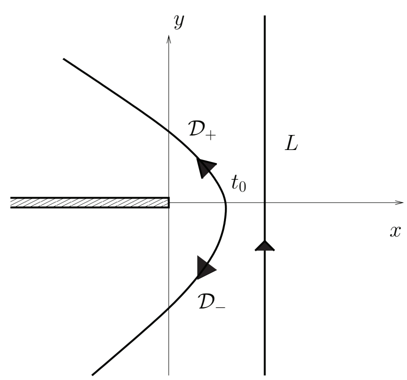


Figure 6: The steepest descent paths $\mathcal{D}_+$ and $\mathcal{D}_-$ emanating from $t = 1/4a^2$.

$$I(\lambda) = \frac{1}{2\pi i}\left(\int_{\mathcal{D}_+} - \int_{\mathcal{D}_-}\right)\frac{e^{\lambda(-\frac{1}{4a}-W)}}{t}\frac{dt}{dW}\,dW.$$

Now

$$\frac{dt}{dW} = \sigma(1+\sigma 2i\sqrt{aW})\frac{i}{2a^{\frac{3}{2}}}W^{-\frac{1}{2}},$$

where $\sigma = 1$ for $\mathcal{D}_+$ and $\sigma = -1$ for $\mathcal{D}_-$.

Also

$$\frac{1}{t} = 4a^2(1+2i\sigma\sqrt{aW})^{-2}.$$

To use Watson's lemma we need the expansion of $\frac{1}{t}\frac{dt}{dW}$ as $W \to 0+$. Using the above expression gives

$$\frac{1}{t}\frac{dt}{dW} = \frac{4a^2 i}{2a^{\frac{3}{2}}}(\sigma W^{-\frac{1}{2}} + 2ia^{\frac{1}{2}})(1+2i\sigma a^{\frac{1}{2}}W^{\frac{1}{2}})^{-2},$$

$$\sim 2ia^{\frac{1}{2}}[\sigma W^{-\frac{1}{2}} - 2ia^{\frac{1}{2}} - 4a\sigma W^{\frac{1}{2}} + 8ia^{\frac{3}{2}}W + 16a^2\sigma W^{\frac{3}{2}} + \ldots].$$

Here we have used the fact that $\sigma^2 = 1$. Thus

$$I(\lambda) \sim \frac{1}{2\pi i}2a^{\frac{1}{2}}ie^{-\frac{\lambda}{4}}\Big[$$

$$\int_0^\infty e^{-\lambda W}(W^{-\frac{1}{2}} - 2ia^{\frac{1}{2}} - 4aW^{\frac{1}{2}} + 8ia^{\frac{3}{2}}W + 16a^2W^{\frac{3}{2}} + \ldots)\,dW$$

$$+\int_0^\infty e^{-\lambda W}(W^{-\frac{1}{2}} + 2ia^{\frac{1}{2}} - 4aW^{\frac{1}{2}} - 8ia^{\frac{3}{2}}W + 16a^2W^{\frac{3}{2}} + \ldots)\,dW\Big],$$

Hence

$$I(\lambda) \sim 2a^{\frac{1}{2}}\frac{e^{-\frac{\lambda}{4}}}{\pi}\left[\frac{\Gamma(\frac{1}{2})}{\lambda^{\frac{1}{2}}} - 4a\frac{\Gamma(\frac{3}{2})}{\lambda^{\frac{3}{2}}} + 16a^2\frac{\Gamma(\frac{5}{2})}{\lambda^{\frac{5}{2}}} + \dots\right],$$

$$\sim 2\sqrt{\frac{a}{\pi\lambda}}e^{-\frac{\lambda}{4a}}\left[1 - \frac{2a}{\lambda} + \frac{12a^2}{\lambda^2} + \dots\right],$$

as $\lambda \to \infty$.

Example Consider

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt. \tag{7.1}$$

Now

$$p(t) = i(t + t^3/3), \quad p'(t) = i(1+t^2), \quad p''(t) = 2it.$$

Hence we have simple saddle points at $t = \pm i$ and

$$p(\pm i) = \mp 2/3, \quad p''(\pm i) = \mp 2.$$

Thus the directions of steepest descent from $t = i$ are $\theta = 0, \pi$ and the directions of steepest descent from $t = -i$ are $\theta = \pi/2, 3\pi/2$.

Next note that if we set $t = Re^{i\phi}$ then for large R and $\lambda > 0$,

$$e^{i\lambda(t+t^3/3)} \sim O(e^{-\frac{\lambda R^3}{3}\sin(3\phi)})$$

and this decays provided the $\sin(3\phi)$ term is positive. So if we displace the contour in the upper-half plane the contour should begin and end in the sectors

$$2\pi/3 < \phi < \pi, \quad \text{and} \quad 0 < \phi < \pi/3.$$

The steepest descent/ascent paths satisfy

$$\Im(p(t)) = \Im(p(\pm i)) = 0$$

giving with $t = x + iy$,

$$\Im[i(x + iy + \frac{1}{3}(x^3 + 3ix^2y - 3xy^2 - iy^3))] = x(1 + \frac{x^2}{3} - y^2).$$

So the steepest descent paths emanating from $t = i$ are $1 + \frac{x^2}{3} - y^2 = 0$ and from $t = -i$ are $x = 0$.

Note also that if $y^2 = (\frac{x^2}{3} - 1)$ then for large x we have $y \sim \pm\frac{1}{\sqrt{3}}x$. A sketch of the path is given in fig. 7.

The above analysis suggests that we can deform the original contour in (7.1) to the upper-half plane on to the steepest decsent path through $t = i$, see fig. 7. Applying Cauchy"s theorem we obtain

$$\int_{L_1+C_1+C_2+C_3} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt = 2\pi i Res[t = i/\sqrt{2}],$$

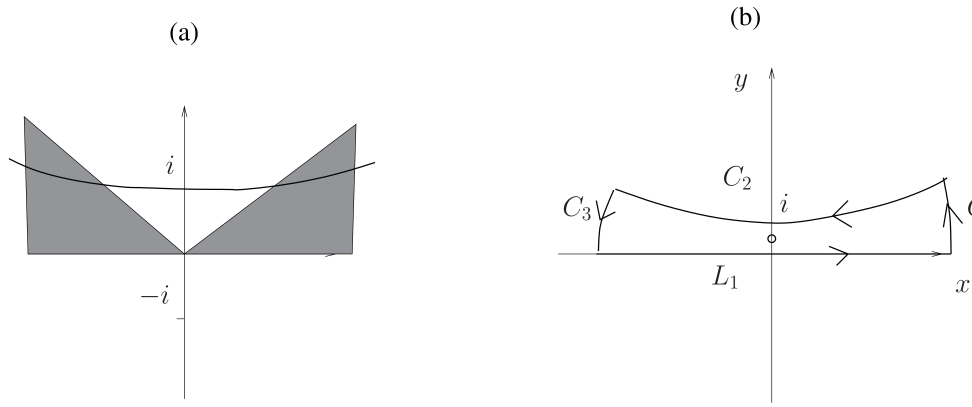


Figure 7: (a) Steepest descent path through $t = i$ (b) Contours for application of Cauchy's theorem.

since the integrand has a simple pole at $t = i/\sqrt{2}$. The integrals along C_1, C_3 goes to zero for large R and so

$$\int_{L_1} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt = \int_{-C_2} \frac{e^{i\lambda(t+t^3/3)}}{2t^2+1}\,dt + 2\pi i[\frac{e^{i\lambda(\frac{i}{\sqrt{2}}-\frac{i}{6\sqrt{2}})}}{\frac{4i}{\sqrt{2}}}].$$

For the integral along the steepest descent path we can put (for the leading order contribution only) $t = i + T$ to obtain

$$I(\lambda) \sim \int_{-\infty}^{\infty} \frac{e^{\lambda(-\frac{2}{3}-T^2)}}{(-2+1)}\,dT + \frac{\sqrt{2}}{2}\pi e^{-\frac{5\lambda}{6\sqrt{2}}},$$

$$\sim -\sqrt{\frac{\pi}{\lambda}}e^{-2\lambda/3} + \frac{\sqrt{2}}{2}\pi e^{-\frac{5\lambda}{6\sqrt{2}}}.$$