Relations, Sentences, Definable Sets

Here are the three examples of binary relations from the previous set of questions.

a.(i) $\mathcal{X} = (X, R^{\mathcal{X}})$ where $X = \{2, 3, 4\}$ and $R^{\mathcal{X}}(x, y)$ means $x < y$.

a.(ii) $\mathcal{X} = (X, R^{\mathcal{X}})$ where $X = \{2, 3, 4, 5, 6, 7, 8\}$ and $R^{\mathcal{X}}(x, y)$ means $x|y$ ("x divides y" in $\mathbb{Z}$, that is there is an integer z, not necessarily in X, such that $y = xz$).

a.(iii) $\mathcal{X} = (X, R^{\mathcal{X}})$ where $X = \{2, 3, 4, 5, 6, 7, 8\}$ and $R^{\mathcal{X}}(x, y)$ means x and y have the same prime factors (but not necessarily with the same multiplicities).

1a. In which of these structures $(X, R^{\mathcal{X}})$ is $R^{\mathcal{X}}$ an equivalence relation? a partial order?

1b. Which of the above structures $(X, R^{\mathcal{X}})$ satisfies the sentence:

(i) $\exists x \, \forall y \, (\neg R(x, y) \wedge \neg R(y, x))$;

(ii) $\exists x, y \, (x \neq y \wedge \exists z \, (\neg R(z, x) \wedge \neg R(z, y)))$?

1c. For each of these structures $\mathcal{X}$, determine the set $\phi(\mathcal{X})$, where ϕ is the formula (with free variable x):

(i) $\exists y \, \exists z \, (R(z, y) \wedge R(y, x))$;

(ii) $R(x, x) \rightarrow \forall y \, R(x, y)$.

Relations, Sentences, Definable Sets, with Answers

Here are the three examples of binary relations from the previous set of questions.

a.(i) $\mathcal{X} = (X, R^{\mathcal{X}})$ where $X = \{2, 3, 4\}$ and $R^{\mathcal{X}}(x, y)$ means $x < y$.

a.(ii) $\mathcal{X} = (X, R^{\mathcal{X}})$ where $X = \{2, 3, 4, 5, 6, 7, 8\}$ and $R^{\mathcal{X}}(x, y)$ means $x|y$ ("x divides y" in $\mathbb{Z}$, that is there is an integer z, not necessarily in X, such that $y = xz$).

a.(iii) $\mathcal{X} = (X, R^{\mathcal{X}})$ where $X = \{2, 3, 4, 5, 6, 7, 8\}$ and $R^{\mathcal{X}}(x, y)$ means x and y have the same prime factors (but not necessarily with the same multiplicities).

1a. In which of these three structures $(X, R^{\mathcal{X}})$ is $R^{\mathcal{X}}$ an equivalence relation?
- **a.**(iii)
a partial order? **a.**(ii) (and **a.**(i) is a strict partial order, meaning like a partial order but like $<$ rather than $\leq$)

1b. Which of the above structures $(X, R^{\mathcal{X}})$ satisfies the sentence:

(i) $\exists x\, \forall y\, (\neg R(x, y) \wedge \neg R(y, x))$; roughly "there is an element ("x") which is not connected to (neither pointing to nor pointed at by) any point (including itself!)
None of them satisfies this.

(ii) $\exists x, y\, (x \neq y \wedge \exists z\, (\neg R(z, x) \wedge \neg R(z, y)))$: roughly "there are two different elements ("x, y") and some element ("z") that points to neither of them"
All of them satisfy this sentence.

1c. For each of the above structures $\mathcal{X}$ determine the set $\phi(\mathcal{X})$, where ϕ is the formula (with free variable x):

(i) $\exists y\, \exists z\, (R(z, y) \wedge R(y, x))$; roughly "there's an element ("y") which points to x and which is itself pointed to (by "z"); these could all, for instance, be the same element "x" if it, "x", points to itself
a.(i) $\{4\}$; **a.**(ii) X; **a.**(iii) X.

(ii) $R(x, x) \rightarrow \forall y\, R(x, y)$. roughly "if x points to itself then x points to every element" Note that if an element doesn't point to itself then it does satisfy this condition (remember the truth table for "$\rightarrow$").
a.(i) X; **a.**(ii) $\emptyset$; **a.**(iii) $\emptyset$.