1 Ordinary Differential Equations—Separation of Variables

1.1 Introduction

Calculus is fundamentally important for the simple reason that almost everything we study is subject to change. In many if not most such problems, the problem is modeled by an equation that involves derivations. Such an equation is called a differential equation.

Differential equations take many forms but one of the simplest examples is

\[ \frac{dy}{dx} = 6x. \]

The equation is formed using two variables \( x \) and \( y \). The variable \( x \) is known as the independent variable and the variable \( y \) as the dependent variable.

The aim is to get an equation for \( y \) in terms of \( x \), i.e. of the form \( y = f(x) \); which of course can be solved by integration:

\[
\frac{dy}{dx} = 6x \\
\int dy = \int 6x \, dx \\
y = 3x^2 + C.
\]

Therefore the general solution of \( \frac{dy}{dx} = 6x \) is \( y = 3x^2 + C \) where \( C \) is an arbitrary constant.

Hence we need a boundary condition (typically in the form of an initial condition \( y(0) = \text{something} \)) in order to obtain a unique solution.

For example, suppose that we specify the boundary condition that \( y = 4 \) when \( x = 1 \), written \( y(1) = 4 \).
Then \( 4 = 3(1) + c \Rightarrow c = 4 - 3 = 1 \). Therefore we get the unique solution \( y = 3x^2 + 1 \).

Before we look at different types of differential equations (DE), we introduce some terminology.

Order

The order of a DE is the order of the highest derivative in the equation:

For example, give the order of the following DE’s:

(i) \( \frac{dy}{dx} = 2y \)

(ii) \( \frac{d^2y}{dx^2} - \left( \frac{dy}{dx} \right)^4 + y = 0 \)
The DE’s are:

(i) First order;

(ii) Second order;

(iii) Fourth order.

Linear

A DE is said to be linear if the dependent variable and its derivatives occur to the first power only and if there are no products involving the dependent variable and/or its derivatives.

Example. Which of the following DEs are linear?

a) \( \frac{dy}{dx} = x^2 \)  
   Yes.

b) \( \frac{dy}{dx} + 2y = \cos x \)  
   Yes.

c) \( y \frac{dy}{dx} = x^3 \)  
   No.

d) \( \frac{dy}{dx} + 4y^2 x = \sin x \)  
   No.

e) \( \sin x \frac{dy}{dx} + y \cos x = \sin x \)  
   Yes.
1.2 First-order ODEs (Ordinary differential equations)

These are differential equations involving just one variable and its derivatives, such as,

\[
(a) \frac{dy}{dx} = y, \quad (b) \frac{dy}{dx} = 2x + 4 \quad (c) \frac{dy}{dx} = 2 \cos 2x \quad \text{and} \quad (d) \frac{dy}{dx} = 2 \cos 2x + y^2.
\]

The examples (a,b,c) are all fairly straightforward to solve, although there is a slight twist in (a), so I will leave that one until later. We will also see how to solve examples like (d) later in this course.

(b) \[ \frac{dy}{dx} = 2x + 4. \]

By integration \[ \int dy = \int (2x + 4) \, dx \] and so \[ y = x^2 + 4x + A. \]

(c) \[ \frac{dy}{dx} = 2 \cos 2x. \]

By integration \[ \int dy = \int 2 \cos 2x \, dx. \] Therefore \[ y = 2 \left[ \frac{1}{2} \sin(2x) \right] + c \] and hence \[ y = \sin(2x) + c. \]

In more complicated examples it might be necessary to use substitution in the integration step.

For example, suppose that \[ \frac{dy}{dx} = \cos x \sin^2 x \] and we have the boundary condition that \[ y \left( \frac{\pi}{2} \right) = \frac{4}{3}. \]

Then by integration,

\[
\int dy = \int \cos x \sin^2 x \, dx
\]

Use the substitution \[ u = \sin x. \] Then \[ \frac{du}{dx} = \cos x, \] and so \[ \int dy = \int u^2 du \] from which we get \[ y = \frac{1}{3} u^3 + C. \]

It is then necessary to rewrite the equation in terms of \[ x. \] Therefore \[ y = \frac{1}{3} \sin^3 x + C. \]

But \[ y = \frac{4}{3} \] when \[ x = \frac{\pi}{2}. \] Thus \[ \frac{4}{3} = \frac{1}{3} \sin^3 \left( \frac{\pi}{2} \right) + C, \] and so,

\[
\frac{4}{3} - \frac{1}{3} = C \quad \Rightarrow \quad C = 1.
\]

Hence the unique solution is \[ y = \frac{1}{3} \sin^3 x + 1. \]
1.3 Separation of variables

First order ODEs (and higher order ODEs for that matter) fall into various categories. Separation of variables forms a general category which are straightforward to solve.

For separation of variables we require that the equation can be (re)written in the form:

\[ \frac{dy}{dx} = f(x)g(y) \]

where \( f(x) \) is only a function of \( x \) and \( g(y) \) is only a function of \( y \).

Then the general solution of the first order ODE

\[ \frac{dy}{dx} = f(x)g(y) \]

is given by

\[ \int \frac{1}{g(y)} \, dy = \int f(x) \, dx. \]

Often we need to rewrite the equation so that it is in the correct form.

For example, the ODE

\[ y \frac{dy}{dx} - 3 = x \]

can be rewritten as \( y \frac{dy}{dx} = x + 3 \) and then as

\[ \frac{dy}{dx} = \frac{x + 3}{y} = f(x)g(y) \]

where \( f(x) = x + 3 \) and \( g(y) = \frac{1}{y} \).

Therefore the equation \( y \frac{dy}{dx} - 3 = x \) can be solved using separation of variables.

Examples. Which of the following ODEs can be solved by separation of variables?

a) \( \frac{dy}{dx} = \cos x \sin y \) \quad Yes.

b) \( \sin y \times \frac{dy}{dx} + x^2 = 0 \) \quad Yes since \( \frac{dy}{dx} = -\frac{x^2}{\sin y} \).

c) \( \frac{dy}{dx} = x^2 + y \) \quad No.
General Approach

The general approach to solving a first order ODE using separation of variables is as follows:

a) Rewrite (if necessary) the equation in the required form:

\[
\frac{dy}{dx} = f(x)g(y)
\]

b) Find the general solution for

\[
\int \frac{1}{g(y)} \, dy = \int f(x) \, dx
\]

c) If boundary conditions are given, solve to find the unique solution.

Example. Solve the ODE

\[
\frac{1}{y^{2}} \frac{dy}{dx} + x^{2} = 0
\]

subject to the initial condition \(y(0) = 2\).

a) \( \frac{dy}{dx} = -x^{2}y^{2} \).

b) Thus \( \int y^{-2} \, dy = \int -x^{2} \, dx \) and so \( -y^{-1} = -\frac{1}{3}x^{3} + c \).

c) Use the boundary condition \( y(0) = 2 \), to obtain \( c = -\frac{1}{2} \) and hence

\[
y^{-1} = \frac{1}{3}x^{3} + \frac{1}{2}.
\]

It is quite often the case (as is true here) that one has an implicit function of \( y \) rather than an explicit one. We could rewrite the solution as \( y = \frac{1}{\frac{1}{3}x^{3} + \frac{1}{2}} \) but I do not think this is any nicer than the previous expression.
Here is a particular ODE that turns up a lot:

1.3.1 Important Fact

The ODE \( \frac{dy}{dt} = k(y - b) \) has general solution

\[
y = b + Ce^{kt}
\]

where \( C \) is a constant that can take any real value.

**Reason:** There is a slight subtlety here so let us work it out carefully. As usual we can separate variables and get

\[
\int \frac{dy}{y - b} = \int k\,dt,
\]

from which \( \ln|y - b| = kt + B \), for some constant \( B \) and hence \( |y - b| = e^{kt+B} \). Therefore,

\[
y - b = \pm e^{kt+B} = \pm e^B e^{kt} = Ce^{kt},
\]

where \( C = \pm e^B \) can take any real value. QED

**Exercise:** Check that the function \( y = b + Ce^{kt} \) does indeed satisfy \( \frac{dy}{dt} = k(y - b) \).

It is illustrative to see what happens to our solution \( y \) as the parity of \( k \) and the value of \( C \) are varied.

The relevant sketches appear on the next page.
Solution: $\frac{dy}{dt} = ky - b$  (thus $y = b + Ce^{kt}$)

**Case 1**  \(k < 0\)

- \(C > 0\)
- \(C < 0\)

**Case 2**  \(k > 0\)

- \(C > 0\)
- \(C = 0\)
- \(C < 0\)

In **Case 1**, \(y = b\) is a "stable equilibrium" meaning that if you change \(y\) slightly it returns to \(y = b\).

In **Case 2**, \(y = b\) is an "unstable equilibrium" in the sense that if I change \(y\) slightly to \(b + \varepsilon\) then however small but positive \(\varepsilon\), as time increases \(y\) shoots off to \(\pm \infty\).
1.4 Newton’s Law of Cooling

Problem. Sherlock Holmes finds a body at 1am with temperature 30°C. An hour later the body has temperature 25°C. If the room temperature is 10°C, when did the person die?

The basic fact we need to solve this problem is:

Newton’s Law of Cooling: The rate of change of temperature of a body is proportional to the difference between the temperature of the body and the ambient temperature.

So, returning to our problem, we let \( T \) denote the temperature of our body at time \( t \). From Newton’s Law of Cooling 1.4, we get the rate of change in \( T \); that is \( \frac{dT}{dt} \), is proportional to \( T - 10 \). Written mathematically:

\[
\frac{dT}{dt} = k(T - 10) \quad \text{for some constant } k. \tag{1.1}
\]

As an aside, note that here \( k \) will be negative since the temperature will decrease. You could also write \( \frac{dT}{dt} = -k(T - 10) \), with \( k > 0 \). It obviously does not matter which way we do it!

So, now I can apply (1.3.1) to (1.1) to get

\[
y = 10 + Ce^{kt}.
\]

Now I should decide my units for \( t \). Certainly I should measure \( t \) in hours, since that is the question is naturally phrased. More importantly, it is best to take \( t = 0 \) to be 1am, since that will make it easiest to apply our initial conditions. Thus, at \( t = 0 \) we have \( T = 30 \) and so \( 30 = 10 + Ce^0 \) or \( C = 20 \). Thus \( y = 10 + 20e^{kt} \). Next we have \( T(1) = 25 \) from which \( 25 = 10 + 20e^k \) or \( e^k = 15/20 \) and \( k = -0.29 \); thus

\[
y = 10 + 20e^{-0.29t}.
\]

Note that we have indeed found that \( k \) is negative, which fits with our intuition and suggests we are on the right track.

Now finally we can solve the problem: the person died when the body temperature was 37; thus when \( 37 = 10 + 20e^{0.29t} \). In other words \( e^{0.29t} = 27/20 \) and \( t = \frac{\ln(27/20)}{0.29} = -1.03 \).

In other words the person died at time \( t = -1.03 \) or (just before) midnight.

Another example—interest payments: Suppose that you are paying interest on your student loan at a rate of 5%pa, compounded continuously (where pa means per year). So, if the amount of the loan is
\( y(t) \) at time \( t \) (in years) then \( \frac{dy}{dt} = \frac{5}{100}y \). (Do you see why this is true?) This has solution \( y = Ce^{t/20} \).

In other words, the amount you owe grows exponentially.

Now let's make the question harder.

**Question:** Again you are paying 5% interest, compounded continuously but suppose you also pay it off at a continuous rate of £500pa. If you took out a loan of £3,000 how quickly will you pay it off?

**Answer:** Now there are two changes in \( y \). As before, you are paying interest at a rate of 5% which gives a contribution to \( \frac{dy}{dt} \) of \( \frac{1}{20}y \). But now you are also decreasing \( y \) by 500 each year. Thus

\[
\frac{dy}{dt} = \frac{1}{20}y - 500.
\]

If we rewrite this in the form \( \frac{dy}{dt} = \frac{1}{20}(y - 10,000) \), then we can apply (1.3.1) and we get

\[
y = 10,000 + Ce^{t/20}.
\]

From \( y(0) = 3000 \) we get \( C = -7,000 \) and so

\[
y = 10,000 - 7,000e^{t/20}.
\]

Finally the loan is paid off when \( y = 0 \) or \( e^{t/20} = 10/7 \), which gives \( t = 20\ln(10/7) = 7.13 \). So, you pay off in 7.13 years.

You can of course repeat this question for different values of the amount \( y(0) \) you borrow. The sketches are given on the next page. For \( y(0) < 10,000 \) it decreases exponentially, but for \( y(0) > 10,000 \) it increases exponentially.
Interest on your loan amount owed

\[ \begin{align*}
10,000 & \\
3,000 & \rightarrow t
\end{align*} \]

Population growth for different values of \( y(0) \)

\[ \begin{align*}
500 & \\
1,000 & \quad (\text{unstable}) \quad \sin y = 0 \\
& \quad \text{for } y(0) = 0
\end{align*} \]
Example: Population density. (i) First a rather general question. Consider the population density $y(t)$ of a certain population of animals at time $t$. The rate of change of $y(t)$ depends upon two constraints: First the excess of birth rate over death rate; this is proportional to the number of animals present. Second extra deaths due to overcrowding is proportional to the square of the number of animals present. Write down a differential equation that models this.

Answer: Note that the number of animals is proportional to the density of animals, so the question tells us that the birth rate is also proportional to $y(t)$; this gives a contribution to $\frac{dy}{dt}$ of the form $\alpha y$ for some $\alpha$. Similarly the excess death rate gives a contribution to $\frac{dy}{dt}$ of the form $\beta y^2$ for some $\beta$. Thus the equation we want is

$$\frac{dy}{dt} = \alpha y + \beta y^2.$$

Comment: We actually know that $\alpha > 0$ (since extra births increase the population) and $\beta < 0$ (since extra deaths decrease the population. Fortunately we do not need to put in $\pm$ signs as they will always come out in the wash.

(ii) A more explicit version: Suppose in the above equation that $\alpha = 1,000$, $\beta = -1$ and $y(0) = 500$. Find a formula for the population density $y(t)$ and sketch your solution.

Answer: We now have the equation $\frac{dy}{dt} = 1,000y - y^2$ and we can separate variables to give

$$\int \frac{dy}{1,000y - y^2} = \int dt. \quad (1.2)$$

To solve the LHS we need to use partial fractions; so write

$$\frac{1}{1,000y - y^2} = \frac{1}{y(1,000 - y)} = \frac{A}{y} + \frac{B}{1000 - y} = \frac{A(1000 - y) + yB}{y(1000 - y)}.$$

From this we obtain $1 = 1000A - yA + yB$ and so $A = 10^{-3} = B$. Thus

$$\int \frac{dy}{1,000y - y^2} = \int \frac{10^{-3}}{y} dy + \int \frac{10^{-3}}{1000 - y} dy = 10^{-3} \ln |y| - 10^{-3} \ln |10^3 - y| = 10^{-3} \ln \left| \frac{y}{1000 - y} \right|.$$

Hence the solution to (1.2) is

$$10^{-3} \ln \left( \frac{y}{1000 - y} \right) = \int dt = t + C,$$

for some constant $C$. Substituting in $y(0) = 500$ gives $C = 10^{-3} \ln(500/500) = 0$. Therefore,

$$10^{-3} \ln \left( \frac{y}{1000 - y} \right) = \int dt = t.$$
Equivalently, \[ \frac{y}{1000-y} = e^{1000t} \]. If you want you can solve this equation for \( y \), giving
\[
y = \frac{1000 \cdot e^{1000y}}{1 + e^{1000y}},
\]
but I do not think that this is much nicer. The sketch (for a range of different initial conditions) is given on page 10.

**Exercise:** Solve the equation from part (i) of the Population Density Question, for \( \beta = -1 \) but arbitrary \( \alpha \) and arbitrary initial conditions. You should find that \( y = \frac{\alpha A e^{\alpha t}}{1 + A e^{\alpha t}} \). Here, \( A \) will depend upon the initial conditions. Whatever they are, you will see that \( y(t) \to \alpha \) as \( t \to \infty \).

So the same rough sketch applies as for the explicit case.

**Example:** Suppose that the height of a wave satisfies the following ODE:
\[
\frac{dy}{dt} = -ky \cos(t)
\]
where \( k > 0 \).

Suppose that initially (i.e. at time \( t = 0 \)) the wave is 2 units high and at time \( t = \frac{\pi}{2} \) the wave is 1 unit high. Find an expression for the height of the wave for all time.

Solution. Separation of variables gives
\[
\int \frac{dy}{y} = \int -k \cos t \, dt
\]
Thus \( \ln(y) = -k \sin t + c \) which has solution
\[
y = \exp(-k \sin t + c) = e^c \exp(-k \sin t).
\]
At time \( t = 0 \), \( \sin t = 0 \). Therefore \( 2 = e^c e^{-k(0)} \) and so, \( 2 = e^c \). Thus
\[
y(t) = 2 \exp(-k \sin t)
\]
Next at time \( t = \frac{\pi}{2} \), \( \sin t = 1 \). Therefore \( 1 = 2e^{-k} \) and so, \( \frac{1}{2} = e^{-k} \) which can be solved to give \( k = -\ln \left( \frac{1}{2} \right) \).

Thus \( k = \ln(2) = 0.693 \) and the unique solution for the wave is
\[
y(t) = 2 \exp(-0.693 \sin t).
\]