1. (a) Obtain 2-term asymptotic expansions of the roots of
\[(x - 2)^3 - \epsilon(3x + 1) = 0\]
as \(\epsilon \to 0\).

(b) Obtain the 3-term asymptotic expansion of the root of
\[x - \epsilon^2 x \log x = \epsilon\]
as \(\epsilon \to 0\).

(c) Obtain the 2-term asymptotic expansion of the solution of
\[\frac{dx}{dt} = 1 + \frac{\epsilon}{(1 - x)^2}, \quad x(0) = 0.\]
Comment briefly on the nature of the expansion near \(t = 1\).

2. Explain briefly what are the "transition curves" for the equation
\[\ddot{x} + (\delta + \epsilon \phi(t))x = 0,\]
where $\phi(t)$ is periodic with period $T$, stating clearly how these curves are characterised.

For the equation
\[ \ddot{x} + (\delta + 4\epsilon \cos^3 t)x = 0 \]
you are given that a pair of transition curves passes through $\delta = 1$ when $\epsilon = 0$. Show that, when $0 < \epsilon \ll 1$, the curves are given by the approximate formulas
\[ \delta = 1 + \frac{63}{20}\epsilon^2 ..., \]
\[ \delta = 1 - \frac{7}{20}\epsilon^2 ... . \]

3. The function $y(x)$ satisfies the equation
\[ \frac{d^2y}{dx^2} + \{\lambda^2(1 + x^2)^2 + \lambda\}y = 0, \quad y(0) = y(1) = 0. \]

Use the WKB method to obtain asymptotic approximations to the solutions of the differential equation as $\lambda \to \infty$.

Deduce that the large eigenvalues are given by the approximate formula
\[ \lambda_n \approx \frac{3}{4}(n - \frac{1}{8})\pi. \]

4. (a) A non-linear oscillator satisfies the equation
\[ (1 + \epsilon \dot{x}^2)\ddot{x} + 4x = 0. \quad (0 < \epsilon << 1). \]

Use Linstedt’s method to obtain a 2-term approximation to the frequency in terms of the amplitude $a$.

(b) Use the energy balance method to show that the oscillator equation
\[ \ddot{x} + \epsilon(|x|^3 - 1)\dot{x} + x = 0 \quad (0 < \epsilon << 1) \]
has a limit cycle of radius $\left(\frac{15\pi}{8}\right)^{1/3}$ approximately.

5. A non-linear oscillator satisfies the equation
\[ \ddot{x} + \frac{1}{3}\epsilon \dot{x}^3 + x + \frac{1}{3}\epsilon x^3 = 0. \]

Use the method of multiple scales to obtain a uniform approximate solution to $O(\epsilon)$, showing that the slowly-varying amplitude $a$ and phase $\beta$ satisfy the equations
\[ \frac{da}{dT} = -\frac{1}{8}a^3; \]
\[ \frac{d\beta}{dT} = \frac{1}{8}a^2; \]
where \( T = \epsilon t \) is the slow time.

Hence obtain the forms of \( a \) and \( \beta \).

**Formulas**

You are given that

\[
\cos^3 t = \frac{3}{4} \cos t + \frac{1}{4} \cos 3t
\]

\[
\sin t \cos^2 t = \frac{1}{4} \sin t + \frac{1}{4} \sin 3t
\]

\[
\sin^2 t \cos t = \frac{1}{4} \cos t - \frac{1}{4} \cos 3t
\]

\[
2 \cos A \cos B = \cos(A - B) + \cos(A + B)
\]

\[
2 \cos A \sin B = \sin(A + B) + \sin(B - A)
\]

\[
2 \sin A \sin B = \cos(A - B) - \cos(A + B)
\]