Math34011 : Examples 8 : Chapter 5

Q1. Use the method of multiple scales to obtain a uniformly convergent asymptotic expansion for the solution to the first order, of

1. \( \ddot{x} + \epsilon \left( \frac{1}{3} \dot{x}^3 - \dot{x} \right) + x = 0 \)

2. \( \ddot{x} + \epsilon (x^4 - 1) \dot{x} + x = 0 \)

[Hint: you are given the solution of \( \frac{dx}{dy} = \frac{1}{2} x \left( 1 - \frac{1}{8} x^4 \right) \) is \( x = \frac{2}{(2 + 16 k e^{-2y})^{1/4}} \)]

3. \( \ddot{x} + \epsilon \dot{x}^3 + x = 0 \).

Q2. Use the method of multiple scales to obtain a uniform first order solution to Examples 7 Q1 parts 1 and 2. Compare your two answers and comment.

Q3. The orbital equation of a planet about the sun is

\[ \frac{d^2u}{d\theta^2} + u = k \left( 1 + \epsilon u^2 \right), \]

where \( u = 1/r \) and \( (r, \theta) \) are polar coordinates, \( k \) is a constant and \( \epsilon << 1 \) represents a relativistic correction to Newton’s theory.

Use the method of multiple scales to solve this to first order in \( \epsilon \), with the initial condition \( u = k(1 + e), \dot{u} = 0 \) at \( \theta = 0 \). Here, \( e \) is the eccentricity. Show that

\[ u_0 = k r \cos \left[ \theta \left( 1 - \epsilon k^2 \right) \right] + k \]

and deduce that the orbit processes by \( 2\pi \epsilon k^2 \) in each year.

This is an observable effect. The procession of the planet Mercury was one of the crucial tests of Einstein’s theory of general relativity.