OPTIMAL PORTFOLIO PROBLEM UNDER LIQUIDITY CONSTRAINT

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Outline

1. Introduction

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2. Statement of the Problem
   - Similarity Reduction

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- In 1969, Merton introduces stochastic calculus into finances and uses the Bellman principle of optimality, this approximation known as stochastic control and he uses HARA utility functions (solutions both numerical and analytical).
- Nowadays, many authors are researching in the Portfolio optimization theory, but we took some of the ideas form F.Longstaff "Optimal Portfolio Choice and the Valuation of Illiquid Securities" (2001)

Some Definitions

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- Portfolio. Distribution of the actual Wealth in the different investment opportunities. Probability vector that contains the percentages of investment on asset(s) and bond(s) (no money leftovers policy).

Optimal portfolio problem

Continuous-time portfolio optimization problem

The continuous-time portfolio problem consists in maximizing the expected utility or terminal wealth over a trading interval $[0; T]$, this is done in a complete market with one risky asset, and one risk-less bond.

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- Show some numerical results and conclusions.

Wealth process

The prices of one bond and one asset are modeled by the processes

$$dB(t) = B(t)r(t)dt \qquad \text{Bond}$$
$$dS(t) = b(t)S(t)dt + \sigma(t)S(t)dZ(t) \qquad \text{Asset}$$

where $Z(t)$ is a standard brownian motion. The **Wealth process** $W(t)$ is defined by

$$W(t) = N(t)S(t) + M(t)B(t)$$

where $N(t)$ is the number of shares hold in the risky asset and $M(t)$ on the risk-less, in this approximation we will assume $r(t) = 0$, hence $B(t) = 1$

Wealth process (cont.)

Since we can sell and buy at any time we have the following dynamics on the number of shares

$$dN(t) = \gamma(t)dt$$

where $\gamma(t)$ is non stochastic. This implies some dynamics on the number of risk-less , which are

$$dM(t) = -\gamma(t)S(t)dt$$

Because the dynamics of the number of risky and risk-less assets cancel out in the wealth equation, the dynamics of the wealth is

$$dW(t) = N(t)dS(t) = N(t)bS(t)dt + N(t)\sigma S(t)dZ(t)$$

Illiquidity

- What is Illiquidity?.
  "Traders view illiquidity as the situation where their ability to buy or sell securities (at any price) is limited or restricted".(Longstaff)
- It will be represented in our model by the constraint

$$\gamma(t) \in [-\gamma_{MAX}, \gamma_{MAX}]$$

where $\gamma_{MAX} > 0$

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Stochastic Control

In the context the Bellman principle of optimality and using the HJB equation, the problem states that we must find the strategy $\gamma(t)$ that gives the maximum expected utility (logarithmic as in Longstaff), which is a problem in three state variables S, M, N and time t.

$$J(S, N, M, t) = \tag{1}$$
$$\max_{\gamma(t)} \left\{ \frac{1}{2}\sigma^2 S^2(t) J_{SS} + S(t) b J_S + \gamma(t) J_N - \gamma(t) S(t) J_M + J_t = 0 \right\}$$

subject to

$$J(S, N, M, T) = \max E[\ln(W(T))] = \max E[\ln(N(T)S(T) + M(T))]$$

Formulation

We use similarity reduction in our equation (1) in order to convert the 3 variables plus time to 2 variables plus time, we put $R = \dfrac{M}{S}$, since we know the "well" behavior in the boundary, that is

$$\ln(W(T)) = \ln(NS + M) = \ln(S(N + R)) = \ln(N + R) + \ln(S)$$

Hence we have the value function
$J(S, N, M, t) = F(N, R, t) + \ln(S)$

Formulation (cont.)

and thus obtain the following derivatives

$$
\begin{aligned}
J_t &= F_t \\
J_N &= F_N \\
J_M &= F_R \frac{1}{S} \\
J_S &= F_R \left(\frac{-M}{S^2} \right) + \frac{1}{S} \\
J_{SS} &= F_{RR} \left(\frac{-M}{S^2} \right)^2 + F_R \left(\frac{2M}{S^3} \right) + \frac{-1}{S^2}
\end{aligned}
\tag{2}
$$

FORMULATION (CONT.)

replacing (2) in (1) and simplifying, the problem turns to

$$F(N,R,t) = \max_{\gamma(t)} \left\{ \frac{1}{2}\sigma^2 R^2 F_{RR} + \left((\sigma^2 - b)R - \gamma(t)\right) F_R + \gamma(t) F_N + F_t - \left(\frac{1}{2}\sigma^2 - b\right) = 0 \right\} \tag{3}$$

with utility function

$$F(N,R,T) = \ln(N(T) + R(T))$$

Boundary Conditions

We have three conditions to define which are $N = 0$, $R = 0$ and $R = R_{MAX}$ (i.e. $R = \infty$).

- If $N = 0$. we have that the whole wealth is in the non-risky asset so that $W \equiv M$ which gives a boundary of $\ln(M)$. which in our model, measuring $\frac{W}{S}$, correspond to $\ln(R)$.
- If $R = 0$, then we must have $M = 0$ and hence all the money is in the risky asset, after some derivation the corresponding condition to our model leads to $\ln(N) + (b - \frac{1}{2}\sigma^2)(T_{MAX} - t)$.

Boundary Conditions (cont.)

- If $R \to \infty$ we shall use the solution for the total wealth. Suppose that the dynamics of the portfolio $p = \frac{N(t)S(t)}{W(t)} = \frac{N}{N+R}$ are constant on time we have

$$dW(t) = W(t)pbdt + W(t)p\sigma dZ_1(t)$$

and, applying Itô integration (between times t and T) to the above equation, we obtain the value function

$$\begin{aligned} J(W,t) &= E_t\left[\ln(W(T))\right] \\ &= \ln(W(t)) + \left(pb - \frac{1}{2}p^2\sigma^2\right)(T-t) \end{aligned}$$

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Boundary Conditions (cont.)

then, using $R = \frac{M}{S}$ in the value of p to obtain the solution, for non-changing p, for the value function F is

$$F(N,R,t) = \ln\left(N(t)+R(t)\right) + \left(\frac{N(t)}{N(t)+R(t)}b - \frac{1}{2}\left(\frac{N(t)}{N(t)+R(t)}\right)^2 \sigma^2\right)(T-t)$$

In this case we could also implement a Neumann boundary which is

$$\left.\frac{\partial F}{\partial R}\right|_{R_{MAX}} = \frac{1}{N(t)+R_{MAX}}$$

.

Numerical formulation (Semi-Lagrangian Crank-Nicolson)

First remember from equation (3) that we want to maximize $\gamma(t)$ for

$$\frac{1}{2}\sigma^2 R^2 F_{RR} + \left((\sigma^2 - b)R - \gamma(t)\right) F_R + \gamma(t) F_N + F_t - \left(\frac{1}{2}\sigma^2 - b\right) = 0$$

Here we will use a Semi-Langrangian approach on the advection 2-dimensional term (depending on R and N)

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial R}\frac{dR}{dt} + \frac{\partial F}{\partial N}\frac{dN}{dt}$$

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Numerical formulation (Semi-Lagrangian Crank-Nicolson) (cont.)

Thus we have the following problem

$$\frac{DF}{Dt} + \frac{\sigma^2 R^2}{2} F_{RR} = K$$

where $K = \frac{\sigma^2}{2} - b$ which should be solved by Crank-Nicolson, so let us discretize such that $0 \leq n\delta R \leq R_{MAX}$, $0 \leq m\delta N \leq N_{MAX}$,

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Numerical formulation (Semi-Lagrangian Crank-Nicolson) (cont.)

and $0 \leq p\delta t \leq T_{MAX}$, for suitable values of n, m and p in the following way (backwards in time)

$$\frac{F^p_{n,m*} - F^{p-1}_{n,m}}{2\delta t} + \frac{\sigma^2 R^2}{4}\left(\frac{F^p_{n+1,m*} - 2F^p_{n,m*} + F^p_{n-1,m*}}{(\delta R)^2}\right) = \frac{K}{2}$$

$$\frac{F^p_{n,m*} - F^{p-1}_{n,m}}{2\delta t} + \frac{\sigma^2 R^2}{4}\left(\frac{F^{p-1}_{n+1,m} - 2F^{p-1}_{n,m} + F^{p-1}_{n-1,m}}{(\delta R)^2}\right) = \frac{K}{2}$$

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Numerical Results

$\sigma = 0.8, b = 0.2, r = 0, R \in [0, 4.2], N \in [0, 4], \delta R = 0.01, \delta N = 0.01, \delta t = 0.001$

	$\gamma_{MAX} = 0$		$\gamma_{MAX} = 0.5$		$\gamma_{MAX} = 0.9$	
N	DB	NB	DB	NB	DB	NB
0.1	0.01458	0.01461	0.02521	0.02526	0.02621	0.02618
0.25	0.02519	0.025222	0.02778	0.0278	0.02799	0.02797
≈ 0.3	0.02573	0.02575	0.02782	0.02784	0.02802	0.02801
0.5	0.01504	0.01504	0.02475	0.02476	0.02571	0.0257
0.75	-0.02788	-0.02788	0.007	0.007	0.01481	0.01481
0.9	-0.07442	-0.07442	-0.01692	-0.01692	$9.263e^{-5}$	$9.163e^{-5}$

TABLE: Comparing different boundaries and liquidity constraints

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Graphs

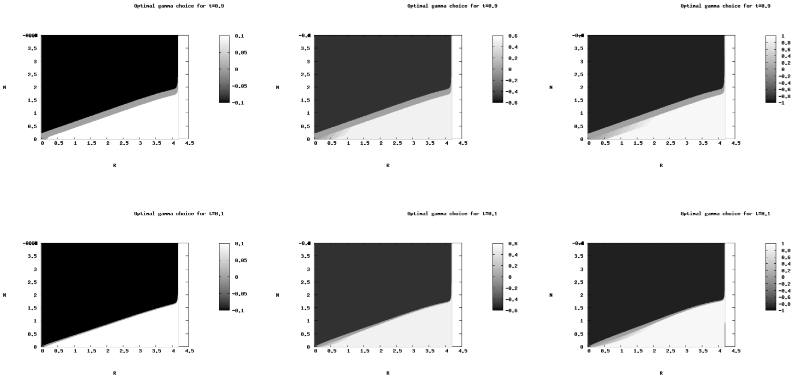

FIGURE: Optimal choices for γ

Conclusions...

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CONCLUSIONS...

- As the constraint moves away from the fully illiquid case, there appear a wider stripe of choices being less aggressive.
- As time elapses, the investors appear to have more aggressive decisions.
- In the Semi-Lagrangian part of the method the linear interpolation performs better than the bicubic interpolation.
- So far, this model has shown the same solutions as Longstaff's models and is simpler.

$$\max_{\gamma(t)} \left\{ \frac{N^2 S^2 \sigma^2}{2} J_{WW} + \frac{S^2 \sigma^2}{2} J_{SS} + N S^2 \sigma^2 J_{WS} + bNSJ_W + bSJ_S + \gamma J_N + J_t = 0 \right\}$$

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Finally

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- Thanks.