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Products of Homogeneous Subspaces in Free Lie Algebras

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Introduction

- Let A be the ring of non-commutative polynomials in indeterminates $X = \{x_1, x_2, \ldots, x_r\}$ with coefficients in a field F. The free Lie algebra over a field F on $X = \{x_1, x_2, \ldots, x_r\}$ is the Lie subalgebra generated by the set X in A with respect to the Lie bracket operation
$$[u, v] = uv - vu,$$
where $u, v \in A$ and it is denoted by L . The cardinality of X is called the rank of L.
- Let L_n denote the degree n homogeneous component of L, that is the subspace spanned by Lie products of degree n in the free generators of L. Thus, the free Lie algebra $L(V)$ is a graded algebra:
$$L = L_1 \oplus L_2 \oplus \ldots \oplus L_n \oplus \ldots$$
- The dimension of L_n for all $n \geq 1$ is given by Witt's formula
$$\dim L_n = f(n, r) = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}},$$
where r is the rank of L and μ is the Möbius function, (see [1, Theorem 5.11]).
- For a subset $Y \subseteq L$ we let $L(Y)$ denote the Lie subalgebra generated by Y in L. We write $L_n(Y)$ for the degree n homogeneous component of $L(Y)$.

Products of homogeneous subspaces in L

In [2], R. Stöhr and M. Vaughan-Lee obtained formulae for the dimensions of the subspaces $[L_m, L_n] \leq L_{m+n}$ for all $m, n \geq 1$. They proved that,

Theorem If $m > n$ and $n \nmid m$, then
$$\dim([L_m, L_n]) = \dim L_m \dim L_n \tag{1}$$
If $m = sn$, $s \geq 1$, then
$$\dim([L_m, L_n]) = (\dim L_m - f(s, \dim L_n)) \dim L_n + f(s + 1, \dim L_n) \tag{2}$$

By generalizing this theorem to products of two homogeneous subspaces, we get

Theorem Let U and V be subspaces of L such that $U \subseteq L_m$, $V \subseteq L_n$ with $m \geq n \geq 1$. Then
$$\dim[U, V] = \dim[U \cap L(V), V] + (\dim U - \dim(U \cap L(V))) \dim V.$$

By applying this theorem with $V = L_m$ and $U = L_n$, we obtain that if $n \nmid m$, then the dimension of the subspace $[L_m, L_n]$ is given by (1), and if $m = sn$ with $s \geq 1$, then the dimension this subspace is given by (2).

Our main result gives formulae for the dimension of subspaces of the form $[L_m, L_n, L_k]$ under various conditions on m, n and k.

Main Result

Theorem Let $[L_m, L_n, L_k]$ be a product of three homogeneous subspaces, where $m \geq n$. Then

(i) If $m + n > k$ but $k \nmid m$ or $k \nmid n$, then
$$\dim[L_m, L_n, L_k] = \dim[L_m, L_n] \dim L_k,$$
(ii) if $m + n > k$, $m = sk$ and $n = tk$ with $s, t \geq 1$, then
$$\dim[L_m, L_n, L_k] = \dim[L_s(L_k), L_t(L_k), L_k] + (\dim[L_m, L_n] - \dim[L_s(L_k), L_t(L_k)]) \dim L_k,$$
(iii) and if $k \geq m + n$ and $m + n \nmid k$,
$$\dim[L_m, L_n, L_k] = \dim[L_m, L_n] \dim L_k,$$
(iv) if $k \geq m + n$ and $k = p(m + n)$, then
$$\dim[L_m, L_n, L_k] = \dim L_{p+1}([L_m, L_n]) + (\dim L_k - \dim(L_p([L_m, L_n]))) \dim[L_m, L_n].$$

Since we have explicit formulae for the dimension of $[L_m, L_n]$ for all m, n, namely formulae (1) and (2), all but one of the terms on the right hand sides of the four formulae in main theorem can be expressed in terms of Witt's dimension function $f(n, r)$. The exception is $\dim[L_s(L_k), L_t(L_k), L_k]$ in the formula in part (ii) of the theorem.

The smallest possible instance of such a product is $[L_2, L_2, L_1]$. It turns out that its dimension depends on the characteristic of the field.

Let L be the free Lie algebra of rank r over a field F. If $r \geq 5$, then the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.

We find that for the free Lie algebra L of rank r over a field F one has

$$\dim[L_2, L_2, L_1] = \begin{cases} \dim[L_2, L_2] \dim L_1, & \text{if } charF \neq 2\,; \\ \dim[L_2, L_2] \dim L_1 - \binom{r}{5}, & \text{if } charF = 2\,, \end{cases}$$

with the convention that $\binom{r}{5} = 0$ for $r < 5$.

References

[1] W. MAGNUS, A. KARRASS, D. SOLITAR, *Combinatorial group theory* ,Interscience Publishers [John Wiley and Sons, Inc.], New York,London, Sidney, (1966).

[2] R. STÖHR, M.VAUGHAN-LEE, *Products of homogeneous subspaces in free Lie algebras*,World Scientific, Vol. 19, No.5, 699-703,(2009).