VIEW FROM THE PENNINES: A COSMIC INDIAN ROPE TRICK

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This year we have had both frogspawn and toadspawn in the pond. The tadpoles emerged first and very quickly started to eat the toadspawn. Since the toad tadpoles are poisonous to many creatures it is not clear whether the tadpoles were eating the developing toad tadpoles or just the jelly, but whichever combination they were eating, the carefully roped toadspawn was rapidly destroyed. It is possible that the frog tadpoles were getting their retaliation in early, and that if they fail to destroy the toad tadpoles before they leave the spawn they might become food for the toads-to-be in their turn. To preserve a few toad tadpoles we removed them from the pond and kept them in a jar. One characteristic I had not appreciated is that whilst frog tadpoles rest horizontally on the bottom of the pond, young toad tadpoles rest vertically with their heads up; a position which would be very unstable in the absence of the supporting relatively high density water. Indeed, I have not discovered the source of the stability of this pose, which should be closer to an inverted pendulum than anything else I can think of, but I suppose that a light inverted pendulum could be stable in treacle, though had I been thinking about it at the time I would have looked to see whether there was any periodic oscillation in the tadpoles' tails.

My thoughts turn to small oscillations because of recent work on stabilizing an inverted pendulum, and even highly articulated multiple pendula such as the bicycle chain, using periodic or random perturbations [1, 2, 6]. Experimentalists and applied mathematicians produced a series of wonderful experiments and applications in the mid 1990s which reproduce the effect of the classic Indian rope trick – a flexible wire can be stabilized in the upwards direction by the right sort of oscillation at the base [5, 7]. This provides a great example of the power of applied mathematics to explain surprises (it took no new physics to understand what was going on, just the time and expertise required to analyze the equations).

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I suppose that the ubiquity of the pendulum equation – simple harmonic motion, the simplest linear systems etc – should have led me to expect unexpected applications of this idea. After all linear systems describe the first order model of local behaviour (normal modes) of any solution. Nonetheless, I was intrigued and surprised by a recent paper suggesting that mass density fluctuations in cosmological models could stabilize relatively unstable (i.e. de-focussing) motion of nearby photons. Here is the Indian rope trick on a cosmic scale!

Dettmann et al [3] consider the motion of photons in a standard expanding model of the universe (a Robertson-Walker model) described by the metric (in units for which $G = c = 1$)

$$ds^2 = R^2(\tau)\left((1+2\Phi)d\tau^2 - \frac{1-2\Phi}{(1+\frac{1}{4}Kr^2)^2}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\ d\phi^2)\right) \tag{1}$$

where τ is a time variable, (r,θ,ϕ) are spatial spherical polar coordinates, R is a scale factor describing the expansion and Φ is the Newtonian gravitational potential of mass in the universe, Φ is small and fluctuates quickly due to the speed at which photons move through the universe. The constant K determines the spatial curvature, so K negative is negative curvature (like moving on a saddle) and K positive corresponds to positive curvature.

If we consider the linearization of the relative motion of two nearby photons in this metric then the equation for the infinitesimal separation perpendicular to the photon direction can be expressed in Cartesian coordinates (x, y) as

$$\begin{aligned} x'' &= -(K+2\Phi_{xx})x - 2\Phi_{xy}y \\ y'' &= -2\Phi_{xy}x - (K+2\Phi_{yy})y \end{aligned} \tag{2}$$

where primes denote differentiation with respect to τ and subscripts denote partial derivatives. This is the geodesic deviation equation for the metric, and in the absence of the Newtonian potential Φ both x and y show oscillations (simple harmonic motion) if $K > 0$ and instability for typical initial conditions if $K < 0$. This corresponds to the focussing of photons in a universe with positive curvature, and the de-focussing effect of the saddle structure in a space with negative curvature. Of course, from a mathematical point of view, the x coordinate and y coordinates of the photon separation is like a pendulum hanging downwards if $K > 0$ and the inverted pendulum if $K < 0$ – this suggests that the work on periodic or stochastic stabilization of the inverted pendulum could be applicable to the full system with $K < 0$

if the derivatives of Φ behave appropriately, the only difference to standard models being that (3) represents two coupled identical pendula rather than the more standard articulated pendulum.

Dettmann et al [3] argue that because the photons move through the mass field so quickly, and because the mass field is so clumpy, it is reasonable to assume that the derivatives Φ_{ij}, $i, j \in \{x, y\}$, of (3) can be modelled by stochastic terms $Af_{ij}(\tau)$ with characteristic frequency ω large, and amplitude A. By splitting the deviations up into slow parts X and Y and highly oscillatory parts x_ω and y_ω, i.e. $x = X + x_\omega$ and $y = Y + y_\omega$, it is possible to integrate out the fast oscillations and obtain an approximation to the slow evolution (X, Y) as

$$\begin{aligned} X'' &= -(K + A^2v^2)X \\ Y'' &= -(K + A^2v^2)Y \end{aligned} \tag{3}$$

where v^2 is an average mean square of the integrated effect of the functions f_{ij}, which will be of order $1/\omega$. Thus these extra terms stabilize the unstable inverted pendulum solutions if $K < 0$ provided $A^2v^2 + K > 0$. Order of magnitude estimates suggest that this is not unreasonable.

This is a beautiful example of a known result having surprising interpretations, and seeing this as a linear stability result rather than specific to the inverted pendulum suggests that the effect might be seen in many other contexts. Dettmann et al [4] use almost exactly the same ideas to look at stabilization due to vacuum fluctuations. As for the toad tadpoles, as they grow they stop resting in the vertical position and adopt the more standard frog tadpole horizontal pose. This might be because as they grow their $-K$ increases until the stability criterion can no longer be satisfied.

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References

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