Fluid Dynamics, Vol. 41, No. 1, 2006, pp. 49–62. Translated from Izvestiya Rossiiskoi Academii Nauk, Mekhanika Zhidkosti i Gaza, No. 1, 2006, pp. 54–67. Original Russian Text Copyright © 2006 by Kudryashov and Chernyavskii.

Nonlinear Waves in Fluid Flow through a Viscoelastic Tube

N. A. Kudryashov and I. L. Chernyavskii

Received March 1, 2005

Abstract — A system of nonlinear equations for describing the perturbations of the pressure and radius in fluid flow through a viscoelastic tube is derived. A differential relation between the pressure and the radius of a viscoelastic tube through which fluid flows is obtained. Nonlinear evolutionary equations for describing perturbations of the pressure and radius in fluid flow are derived. It is shown that the Burgers equation, the Korteweg-de Vries equation, and the nonlinear fourth-order evolutionary equation can be used for describing the pressure pulses on various scales. Exact solutions of the equations obtained are discussed. The numerical solutions described by the Burgers equation and the nonlinear fourth-order evolutionary equation are compared.

Keywords: nonlinear waves, evolutionary equations, Burgers equation, Korteweg-de Vries equation, viscoelasticity, variational approach.

The nonlinear wave processes in fluid flow through a viscoelastic tube simulate processes in the cardiovascular system. The vessels of the vascular system (arteries and arterioles) perform the conducting and damping functions [1–4]. The conducting function is responsible for transporting oxygen-enriched blood to the various organs and tissues while the damping function leads to the smoothing of pressure pulses so that at a certain distance from the aorta the blood flow becomes almost steady-state. Diseases of the cardiovascular system disturb both these functions. For example, atherosclerosis is an impairment of conductivity as a result of cholesterol deposits and luminal narrowing up to occlusion (complete blockage) of the vessel. This leads to ischemic diseases of the tissues. The atherosclerotic process mainly affects the large arteries. Arteriosclerosis is an impairment of the damping function in which the pressure pulses are poorly smoothed due to structural changes in the vessel walls. This leads to an increase in blood pressure (hypertension) and secondary damage to the vessels. Therefore, it is of interest to construct and analyze a model that takes into account the specific properties of the vessel wall.

Many approaches to studying the blood flow in vessels have used linear models [5]. However, already in [6–9] the need to take nonlinear effects into account was noted. In a series of studies linear [10, 11] and nonlinear [12, 13] equations relating the pressure and the radius have been obtained for describing the interaction between the wall and the fluid flow. An analysis of one-dimensional closed systems in the long-wave approximation leads to a set of reduced equations. These include the Burgers [12], Korteweg-de Vries [12, 14, 15], and Korteweg-de Vries-Burgers equations [12]. Many studies have been devoted to the numerical solution of the problem of fluid flow in viscoelastic and elastic tubes using the finite-difference and finite-element methods, for example, [13, 16]. In this case the higher derivatives corresponding to viscoelastic effects are neglected in view of the complexity of the numerical simulation [11]. At the same time, the viscoelasticity of the wall is important in connecting with the damping of high-frequency oscillations in the blood stream [6].

The aim of the present study is to take into account the nonlinear elasticity and viscoelasticity of the wall and extend the family of evolutionary equations to describe the nonlinear wave processes. A variational method is used for deriving the equation relating the pressure and the radius. The use of the multiscale

0015–4628/06/4101–0049

technique makes it possible to separate and classify the effect of the mechanical properties of the system on the evolution of perturbation waves. This approach can also be used to obtain a series of other evolutionary equations corresponding to other mechanical properties of the model.

1. EQUATION RELATING THE PRESSURE AND THE RADIUS OF A VISCOELASTIC TUBE IN FLUID FLOW

We will consider the flow of a fluid, assumed to be incompressible, through an axially symmetric tube under the following assumptions: (1) the tube wall density is constant; (2) the tube strain is characterized by the change in its radius, which depends on the coordinate and time; (3) the deformation of the tube wall and the wall thickness are small as compared with the radius and the characteristic lengths of the wave processes are much greater than the equilibrium radius; and (4) the pressure of the fluid in the flow is the same over the entire tube cross-section and depends on the coordinate and time.

In order to derive the equation of motion of the tube wall we will use the principle of least action in accordance with which the true motion is realized on the extremals of the action

$$J[R(x,\ t)] = \int_{t_0}^{t_1} L\,dt \to \min_{R(x,\ t)} \tag{1.1}$$

We represent the Lagrangian L which characterizes the state of the system as the difference between the kinetic and potential energies

$$L = T - U, \qquad U = U_{el} - A \tag{1.2}$$

where A is the work done by the dissipation and pressure forces in expanding the tube and U_{el} is the elastic potential energy of the tube.

We will consider a fragment of the tube in the cylindrical coordinate system $(r,\ x \equiv z)$. The kinetic energy of a tube element of length l, where l corresponds to the characteristic wave lengths in the fluid flow, can be represented in the form:

$$T = \int_0^l \pi \rho_w h R R_t\,dx$$

where ρ_w is the volume density of the tube wall, h is its thickness, and $R = R(x,\ t)$ is the radius of the tube wall. from the incompressibility of the tube wall and the conservation of its mass there follows the condition $hR = \text{const} = h_0 R_0$, where h_0 and R_0 are the thickness of the undisturbed wall and the equilibrium radius of the tube, respectively. Then the kinetic energy takes the form:

$$T = \int_0^l \pi \rho_w h_0 R_0 R_t\,dx \tag{1.3}$$

The elastic potential energy of a tube element of length l consists of two parts. The first characterizes the elastic energy of the wall as a system of independent nonlinearly elastic rings

$$U_1 = \int_0^l \left[\pi \kappa h (R - R_0)^2 + \frac{2\pi \kappa_1 h}{3}(R - R_0)^3\right] dx$$

$$\kappa = \frac{E}{R_0(1 - \sigma^2)}$$

Here, κ is the linear elasticity coefficient characterizing the extension of a tube element, E is the longitudinal Young's modulus, σ is Poisson's ratio, and κ_1 is the nonlinear elasticity coefficient (quadratic correction to Hooke's law).

The second part is characterized by the elastic energy of the longitudinal wall fibers which is proportional to the increase in the area of a wall element of length l due to bending along the x axis:

$$U_2 = \int_0^l 2\pi khR\sqrt{1 + R_x^2}\,dx - \int_0^l 2\pi khR\,dx \tag{1.4}$$

The coefficient k characterizes the longitudinal wall stresses. In accordance with [11], it is equal to σ_{xx}, where σ_{xx} is the normal axial component of the wall stress tensor. For blood vessels this corresponds to their constant stress along the axis.

From (1.4) for small strains we obtain

$$U_2 = \int_0^l \pi khRR_x^2\,dx$$

The total elastic potential energy of a tube fragment of length l can be determined from the expression

$$U_{el} = \int_0^l \left(\pi\kappa h(R - R_0)^2 + \frac{2}{3}\pi\kappa_1 h(R - R_0)^3 + \pi khRR_x^2\right) dx \tag{1.5}$$

Let the pressure P_e on the outer surface of the tube be constant, while the fluid pressure $P(x, t)$ is assumed to be constant over the vessel cross-section. By analogy with the fluid viscosity, we will take into account the viscous forces of the vessel walls. The elementary work done by these viscous forces [17], the forces resisting the motion of the wall, the fluid pressure forces, and the external pressure forces can be taken into account by means of the formula

$$\delta A = \int_0^l 2\pi R\left(hf - \mu\frac{\partial R}{\partial t}\right)\sqrt{1 + R_x^2}\,dx\,\delta R + \int_0^l 2\pi R(P - P_e)\,dx\,\delta R$$
$$f = \chi\frac{\partial^3 R}{\partial x^2 \partial t} \tag{1.6}$$

where the force f can be determined in terms of the radial component of the viscous-force stress tensor [17].

Simplifying (1.6), we have

$$\delta A = \int_0^l 2\pi R\left[\chi h\frac{\partial^3 R}{\partial x^2 \partial t} - \mu\frac{\partial R}{\partial t} + P - P_e\right] dx\,\delta R \tag{1.7}$$

Here, χ is the viscosity coefficient of the tube material introduced by analogy with the dynamic viscosity of the fluid and μ is the proportionality coefficient of the resistance of the medium to the motion of the tube wall.

Taking expressions (1.2), (1.3), and (1.5) into account, we obtain a Lagrangian in the form:

$$L = \int_0^l \left(\pi\rho_w h_0 R_0 R_t^2 - \pi\kappa h(R - R_0)^2 - \frac{2}{3}\pi\kappa_1 h(R - R_0)^3 - \pi k\,hR\,R_x^2\right) dx + A$$

where A is given by expression (1.7).

Minimizing the functional (1.1) on the class of smooth functions $R(x,\, t)$ considered on the time interval $[t_0,\, t_1]$, in accordance with the principle of least action we obtain the Euler equation and the transversality conditions in the form:

$$R(P - P_e + \chi h R_{txx} - \mu R_t) = \rho_w h_0 R_0 R_{tt} - khRR_{xx} - \frac{kh}{2}R_x^2 + \kappa h(R - R_0) + \kappa_1 h(R - R_0)^2 \quad (1.8)$$

$$\left.\frac{\partial R(x,\, t)}{\partial x}\right|_{\substack{x=0\\x=l}} = 0, \qquad \left.\frac{\partial R(x,\, t)}{\partial t}\right|_{t=t_1} = 0$$

Taking into account small perturbations of the wall radius $\eta(x,\, t)$

$$R(x,\, t) = R_0 + \eta(x,\, t), \quad h = h_0\frac{R_0}{R}, \quad R_0 = \text{const}, \quad h_0 = \text{const}$$

and neglecting the term $kh_0\eta_x^2/2R_0$ and higher-order terms, in the longwave perturbation approximation from Eq. (1.8) we obtain

$$P - P_e = \rho_w h_0 \eta_{tt} - kh_0\eta_{xx} - \chi h_0 \eta_{txx} + \mu\eta_t + \frac{\kappa h_0}{R_0}\eta + \frac{\kappa_2 h_0}{R_0^2}\eta^2$$

$$\kappa_2 \equiv \kappa_1 R_0 - 2\kappa \quad (1.9)$$

In essence, equation (1.9) is the equation of state for the motion of the fluid in the tube. In the simplest steady-state case the fluid pressure in the tube depends linearly on the radius.

2. EQUATION OF FLUID FLOW IN A VISCOELASTIC TUBE

In order to describe fluid flow in an axially symmetric viscoelastic tube of variable cross-section we will use the continuity equation and the axial component of the two-dimensional Navier-Stokes equation

$$\frac{\partial(vr)}{\partial r} + \frac{\partial(ur)}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \nu_0\left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right] \quad (2.1)$$

where v and u are the radial and axial components of the flow velocity, ν_0 is the kinematic viscosity, and ρ is the fluid density.

We will assume that the radial profile of the axial velocity component has the form:

$$u(r,\, x, t) = \frac{s+2}{s}\left[1 - \left(\frac{r}{R}\right)^s\right]u_a(x,\, t), \quad u_a(x,\, t) = \frac{2}{R^2}\int\limits_0^{R(x,\, t)} u(r,\, x,\, t)r\,dr$$

Here, s is the exponent of the profile steepness. As in [18], averaging the fluid mass and momentum conservation equations (2.1) over the tube cross-section, we arrive at the one-dimensional equations

$$\frac{\partial S}{\partial t} + \frac{\partial(Su_a)}{\partial x} = 0 \quad (2.2)$$

$$\frac{\partial u_a}{\partial t} + u_a\frac{\partial u_a}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \nu_0\frac{\partial^2 u_a}{\partial x^2} - 2\nu_0(s+2)\frac{u_a}{R^2} \quad (2.3)$$

Here, $u_a = u_a(x,\ t)$ is the cross-section-average axial velocity component and $S = S(x,\ t)$ is the tube cross-sectional area. In what follows, the subscript of u_a will be omitted.

Since $S(x,\ t) = \pi R(x,\ t)^2$, equation (2.2) can be reduced to the form:

$$R_t + uR_x + \frac{1}{2}Ru_x = 0$$

Taking into account small variations of the tube radius $R = R_0 + \eta$, we obtain the equation

$$\eta_t + \frac{1}{2}R_0 u_x + \frac{1}{2}\eta u_x + u\eta_x = 0 \tag{2.4}$$

In the present study we will restrict our attention to the analysis of nonlinear waves in the longwave approximation for large Reynolds numbers. In blood vessels this approximation holds for large and medium-sized arteries [6, 11].

Thus, for describing the one-dimensional fluid flow through an axially symmetric viscoelastic tube at large Reynolds numbers we have a system of equations in the form:

$$\begin{aligned} &\eta_t + \frac{1}{2}R_0 u_x + \frac{1}{2}\eta u_x + u\eta_x = 0 \\ &u_t + uu_x + \frac{1}{\rho}P_x = 0 \\ &P = \rho_w h_0 \eta_{tt} - kh_0\eta_{xx} - \chi h_0 \eta_{txx} + \mu\eta_t + \frac{\kappa h_0}{R_0}\eta + \frac{\kappa_2 h_0}{R_0^2}\eta^2 + P_e \end{aligned} \tag{2.5}$$

Assuming that the flow pressure is proportional to the radius perturbation and omitting the nonlinear terms in the system of equations for the inviscid fluid, we obtain the simple linearized system for the fluid flow in an elastic tube

$$\eta_t + \frac{R_0}{2}u_x = 0, \qquad u_t + \frac{1}{\rho}P_x = 0, \qquad P = \frac{\kappa h_0}{R_0}\eta + P_e \tag{2.6}$$

System (2.6) can be written in the form:

$$\eta_t + \frac{R_0}{2}u_x = 0, \qquad u_t + \frac{\kappa h_0}{\rho R_0}\eta_x = 0$$

Differentiating the first and second of the equations with respect to x and t, respectively, we obtain the following linear wave equations for the velocity perturbations

$$u_{tt} = \frac{\kappa h_0}{2\rho}u_{xx}$$

A similar equation holds for the radius and pressure perturbations.

For fluid flow through an elastic tube the pressure pulse propagation velocity obtained by Moence and Korteweg has the form:

$$c_0 = \sqrt{\frac{\kappa h_0}{2\rho}} = \sqrt{\frac{Eh_0}{2\rho R_0(1-\sigma^2)}} \tag{2.7}$$

We introduce the dimensionless variables

$$\begin{aligned} &t = \frac{l}{c_0}t', \qquad x = lx', \qquad u = c_0 u', \qquad \eta = \frac{R_0}{2}\eta' \\ &P = P_0 P', \qquad P_0 = P_e \end{aligned} \tag{2.8}$$

In the dimensionless variables the system of flow equations (2.5) has the form (primes omitted):

$$\begin{aligned} &\eta_t + u_x + \frac{1}{2}\eta u_x + u\eta_x = 0 \\ &u_t + uu_x + \frac{1}{\alpha}P_x = 0 \\ &P = \gamma\eta_{tt} - \beta\eta_{xx} + \lambda\eta_t - \delta\eta_{txx} + \alpha\eta + \alpha_1\eta^2 + 1 \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\alpha = \frac{\rho c_0^2}{P_0}, \qquad \beta = \frac{kh_0R_0}{2P_0l^2}, \qquad \gamma = \frac{\rho_w h_0 R_0 c_0^2}{2P_0l^2} \\ &\delta = \frac{\chi h_0 R_0 c_0}{2P_0l^3}, \qquad \lambda = \frac{\mu R_0 c_0}{2P_0 l}, \qquad \alpha_1 = \frac{\kappa_1 h_0 R_0}{4P_0} - \alpha \end{aligned} \tag{2.10}$$

3. NONLINEAR EVOLUTIONARY EQUATIONS FOR DESCRIBING PERTURBATIONS OF THE FLUID FLOW IN A VISCOELASTIC TUBE

Most evolutionary nonlinear equations can be obtained using the multiscale and perturbation techniques which are now widely represented in the literature. Apparently, this technique was first used in [19]. The system of equations (2.9) contains the small parameters

$$\varepsilon \ll 1 \quad \left(\varepsilon_1 = \frac{a_0}{R_0}, \quad \varepsilon_2 = \frac{R_0}{l}, \quad \varepsilon_3 = \frac{h_0}{R_0}\right)$$

where a_0 is the radius perturbation amplitude. For arteries these parameters have the characteristic values 0.1, 0.4, and 0.2, respectively [14, 20]. Since the characteristic velocities of the pressure waves (pulse waves) are high as compared with the flow velocities, in order to study the evolution of the perturbations in the low-amplitude long wave approximation it is convenient to go over to "slow" time variables, having distinguished the direction of wave propagation. As the parameter ε we will take the smallest of the above parameters ($\varepsilon \sim 0.1$). We will then seek the solution of the system of equations using the variables

$$\eta = \varepsilon^2\eta', \qquad u = \varepsilon^p u', \qquad P = 1 + \varepsilon^p P', \qquad p \in N \tag{3.1}$$

$$\begin{aligned} &\xi = \varepsilon^m(x - t), \qquad \tau = \varepsilon^n t, \qquad m, n \in N, \qquad n > m \\ &\frac{\partial}{\partial x} = \varepsilon^m\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} = \varepsilon^n\frac{\partial}{\partial \tau} - \varepsilon^m\frac{\partial}{\partial \xi} \end{aligned} \tag{3.2}$$

Substitution of (3.2) and (3.1) in (2.9), after cancelling the expressions in the first two equations by ε^{m+p} and in the last equation by ε^p, leads to the system of equations

$$\begin{aligned} &\varepsilon^{n-m}\eta'_\tau - \eta'_\xi + u'_\xi + \varepsilon^p\frac{1}{2}\eta' u'_\xi + \varepsilon^p u'\eta'_\xi = 0 \\ &\varepsilon^{n-m}u'_\tau - u'_\xi + \varepsilon^p u'u'_\xi + \frac{1}{\alpha}P'_\xi = 0 \end{aligned} \tag{3.3}$$

$$\begin{aligned} P' = \varepsilon^{2n}\gamma\eta'_{\tau\tau} - \varepsilon^{n+m}2\gamma\eta'_{\tau\xi} + \varepsilon^{2m}(\gamma - \beta)\eta'_{\xi\xi} + \varepsilon^n\lambda\eta'_\tau - \\ \varepsilon^m\lambda\eta'_\xi - \varepsilon^{n+2m}\delta\eta'_{\tau\xi\xi} + \varepsilon^{3m}\delta\eta'_{\xi\xi\xi} + \alpha\eta' + \varepsilon^p\alpha_1\eta'^2 \end{aligned}$$

We will seek the solution of this system in the form of an asymptotic expansion

$$\begin{aligned} &u' = u_1 + \varepsilon^q u_2 + o(\varepsilon^q); \qquad \eta' = \eta_1 + \varepsilon^q\eta_2 + o(\varepsilon^q) \\ &P' = P_1 + \varepsilon^q P_2 + o(\varepsilon^q), \qquad q \in N \end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.3) and equating the coefficients of ε^0, in the zeroth approximation we have the following relations:

$$-\eta_{1\xi} + u_{1\xi} = 0, \qquad -u_{1\xi} + \frac{1}{\alpha}P_{1\xi} = 0, \qquad P_1 = \alpha\eta_1$$

Hence we obtain

$$u_1(\xi,\tau) = \eta_1(\varepsilon,\tau) + \psi(\tau), \qquad P_1(\xi,\tau) = \alpha\eta_1(\xi,\tau) \tag{3.5}$$

where $\psi(\tau)$ is an arbitrary function determined from the boundary conditions for u_1 and η_1.

In deriving the evolutionary equations containing the derivative with respect to τ and the nonlinear term uu_ξ, with allowance for the first approximation ($\sim \varepsilon^q$) we need to set $n - m = p = q$.

In this case we neglect the terms of higher order than $n - m$ ($n \geq m$) and, after eliminating P', obtain system (3.3) in the form:

$$\varepsilon^{n-m}\eta'_\tau - \eta'_\xi + u'_\xi + \varepsilon^p\frac{1}{2}\eta' u'_\xi + \varepsilon^p u'\eta'_\xi = 0$$

$$\varepsilon^{n-m}u'_\tau - u'_\xi + \varepsilon^p u' u'_\xi + \varepsilon^{2m}\frac{\gamma-\beta}{\alpha}\eta'_{\xi\xi\xi} + \eta'_\xi + \varepsilon^{3m}\frac{\delta}{\alpha}\eta'_{\xi\xi\xi\xi\xi} + \varepsilon^p\frac{2\alpha'}{\alpha}\eta'\eta'_\xi = \varepsilon^m\frac{\lambda}{\alpha}\eta'_{\xi\xi}$$

From this we obtain the equation

$$\varepsilon^{n-m}(\eta'_\tau + u'_\tau) + \varepsilon^p\left(u'u'_\xi + \frac{1}{2}\eta' u'_\xi + u'\eta'_\xi + \frac{2\alpha_1}{\alpha}\eta'\eta'_\xi\right) + \varepsilon^{2m}\left(\frac{\gamma-\beta}{\alpha}\eta'_{\xi\xi\xi}\right) + \varepsilon^{3m}\left(\frac{\delta}{\alpha}\eta'_{\xi\xi\xi\xi\xi}\right) = \varepsilon^m\left(\frac{\lambda}{\alpha}\eta'_{\xi\xi}\right) \tag{3.6}$$

Setting $m = 1$ and taking the relation $n - m = p = q$ into account, we will consider three cases for p, q, and n: (1) $p = q = 1$ and $n = 2$; (2) $p = q = 2$ and $n = 3$; and (3) $p = q = 3$ and $n = 4$.

The increase in the parameter n corresponds to the fact that with time the terms with the higher derivatives play the determining role in the mathematical model.

We will now derive the evolutionary equations for describing perturbations in the flow through a viscoelastic tube.

We will begin by considering the first case ($m = p = q = 1$ and $n = 2$). Substituting (3.4) in (3.6) and equating the coefficients of ε^1, we obtain the equation

$$\eta_{1\tau} + u_{1\tau} + u_1u_{1\xi} + \frac{1}{2}\eta_1u_{1\xi} + u_1\eta_{1\xi} + \frac{2\alpha_1}{\alpha}\eta_1\eta_{1\xi} = \frac{\lambda}{\alpha}\eta_{1\xi\xi}$$

hence with allowance for relations (3.5) we arrive at the evolutionary equation

$$\eta_{1\tau} = \left[\left(\frac{5}{4} + \frac{\alpha_1}{\alpha}\right)\eta_1 + \psi(\tau)\right]\eta_{1\xi} = \frac{\lambda}{2\alpha}\eta_{1\xi\xi} - \frac{\psi'(\tau)}{2} \tag{3.7}$$

Here, $\psi(\tau)$ gives a correction to the wave propagation velocity and when $\psi(\tau) \neq \text{const}$ corresponds to a source. The function $\psi(\tau)$ can be determined from relations (3.5).

Let $\psi(\tau) = \eta_1|_{\xi=\xi_0} - u_1|_{\xi=\xi_0} = 0$, then equation (3.7) goes over into the Burgers equation

$$\eta_{1\tau} + \left(\frac{5}{4} + \frac{\alpha_1}{\alpha}\right)\eta_1\eta_{1\xi} = \frac{\lambda}{2\alpha}\eta_{1\xi\xi} \tag{3.8}$$

If we use (2.7) and (2.10), the dimensionless coefficient of Eq. (3.8) can be expressed in terms of the physical parameters of the model. The coefficient of the nonlinear term is determined from the formula

$$\frac{5}{4} + \frac{\alpha_1}{\alpha} = \frac{1}{4} + \frac{R_0}{2}\frac{\kappa_1}{\kappa}$$

Hence it follows that the ratio of the nonlinear and linear elasticities determines the steepness of the nonlinear wave front. If the wall is linearly elastic, i.e., $\kappa_1 = 0$, then $\alpha_1 = -\alpha$ and equation (3.8) takes the form:

$$\eta_{1\tau} + \frac{1}{4}\eta_1\eta_{1\xi} = \frac{\lambda}{2\alpha}\eta_{1\xi\xi}$$

The coefficient of the second derivative in Eq. (3.8) has the form:

$$\frac{\lambda}{2\alpha} = \frac{R_0}{4\rho l c_0}\mu$$

Thus, the wave attenuation is proportional to the resistance coefficient of the medium.

When $\psi(\tau) = \text{const} = \psi_0$ equation (3.7) can be reduced to (3.8) by means of the nondegenerate change of variables

$$\theta = \xi - \psi_0\tau, \qquad \tau' = \tau$$

Taking (3.1), (3.2), (3.4), and (3.5) into account, we can express the solution of the initial system of equations in the form:

$$\eta(x,\, t) = \varepsilon\eta'(\xi,\, \tau) \simeq \varepsilon\eta_1(\xi,\, \tau); \quad u(x,\, t) = \varepsilon u'(\xi,\, \tau) \simeq \varepsilon u_1(\xi,\, \tau) \simeq \varepsilon\eta_1(\xi,\tau)$$

$$P(x,\, t) = \varepsilon P'(\xi,\, \tau) \simeq \varepsilon P_1(\xi,\, \tau) = \varepsilon\alpha\eta_1(\varepsilon,\, \tau), \quad \xi = \varepsilon(x - t), \quad \tau = \varepsilon^2 t$$

We will now consider the second case ($m = 1$, $p = q = 2$, and $n = 3$). Substituting (3.4) in (3.6) and equating the coefficients of ε^2, we obtain the equation

$$\eta_{1\tau} + u_{1\tau} + u_1u_{1\xi} + \frac{1}{2}\eta_1u_{1\xi} + u_1\eta_{1\xi} + \frac{2\alpha_1}{\alpha}\eta_1\eta_{1\xi} + \frac{\gamma-\beta}{\alpha}\eta_{1\xi\xi\xi} = 0 \tag{3.9}$$

Taking (3.5) into account, from (3.9) we arrive at the evolutionary equation

$$\eta_{1\tau} + \left[\left(\frac{5}{4} + \frac{\alpha_1}{\alpha}\right)\eta_1 + \psi(\tau)\right]\eta_{1\xi} + \frac{\gamma-\beta}{2\alpha}\eta_{1\xi\xi\xi} = -\frac{\psi'(\tau)}{2} \tag{3.10}$$

Setting $\psi \equiv 0$, we obtain the Korteweg-de Vries equation:

$$\eta_{1\tau} + \left(\frac{5}{4} + \frac{\alpha_1}{\alpha}\right)\eta_1\eta_{1\xi} + \frac{\gamma-\beta}{2\alpha}\eta_{1\xi\xi\xi} = 0 \tag{3.11}$$

Using (2.7) and (2.10), we can represent the coefficient of the dispersion term in the form:

$$\frac{\gamma-\beta}{2\alpha} = \frac{1}{4}\left(\frac{R_0}{l}\right)^2\left[\frac{h_0}{R_0}\frac{\rho_w}{\rho} - 2(1 - \sigma^2)\frac{\sigma_{xx}}{E}\right]$$

Thus, the dispersion coefficient is determined by the ratio of the wall and fluid densities and the ratio of the longitudinal wall stress to the elasticity modulus of the wall.

An approximate solution of system (2.9) can be determined in terms of the solutions of the Korteweg-de Vries equation by mean of the relations

$$\eta(x,\,t) = \varepsilon^2\eta'(\xi,\,\tau) \simeq \varepsilon^2\eta_1(\xi,\,\tau);$$
$$u(x,\,t) = \varepsilon^2 u'(\xi,\,\tau) \simeq \varepsilon^2 u_1(\xi,\,\tau) \simeq \varepsilon^2\eta_1(\xi,\,\tau)$$
$$P(x,\,t) = \varepsilon^2 P'(\xi,\,\tau) \simeq \varepsilon^2 P_1(\xi,\,\tau) = \varepsilon^2\alpha\eta_1(\varepsilon,\,\tau),$$
$$\xi = \varepsilon(x - t), \quad \tau = \varepsilon^3 t$$

We will now consider the third case ($m = 1,\ p = q = 3$, and $n = 4$). Substituting (3.4) in (3.6) and equating the coefficients of ε^3, we obtain the equation

$$\eta_{1\tau} + u_{1\tau} + u_1 u_{1\xi} + \frac{1}{2}\eta_1 u_{1\xi} + u_1\eta_{1\xi} + \frac{2\alpha_1}{\alpha}\eta_1\eta_{1\xi} + \frac{\delta}{\alpha}\eta_{1\xi\xi\xi\xi} = 0 \tag{3.12}$$

Taking (3.5) into account, from (3.12) we arrive at the evolutionary equation

$$\eta_{1\tau} + \left[\left(\frac{5}{4} + \frac{\alpha_1}{\alpha}\right)\eta_1 + \psi(\tau)\right]\eta_{1\xi} + \frac{\delta}{2\alpha}\eta_{1\xi\xi\xi\xi} = -\frac{\psi'(\tau)}{2} \tag{3.13}$$

Setting $\psi \equiv 0$, we obtain the fourth-order nonlinear evolutionary equation

$$\eta_{1\tau} + \left(\frac{5}{4} + \frac{\alpha_1}{\alpha}\right)\eta_1\eta_{1\xi} + \frac{\delta}{2\alpha}\eta_{1\xi\xi\xi\xi} = 0 \tag{3.14}$$

Using (2.7) and (2.10), we find that the coefficient of the fourth derivative has the form:

$$\frac{\delta}{2\alpha} = \frac{h_0 R_0}{4\rho l^3 c_0}\chi$$

Thus, the attenuation of the amplitude of the wave described by Eq. (3.14) is proportional to the coefficient of viscosity of the tube material.

An approximate solution of the initial system of equations (2.9) can be expressed by the formulas

$$\eta(x,\,t) = \varepsilon^3\eta'(\xi,\,\tau) \simeq \varepsilon^3\eta_1(\xi,\,\tau)$$
$$u(x,\,t) = \varepsilon^3 u'(\xi,\,\tau) \simeq \varepsilon^3 u_1(\xi,\,\tau) \simeq \varepsilon^3\eta_1(\xi,\,\tau)$$
$$P(x,\,t) = \varepsilon^3 P'(\xi,\,\tau) \simeq \varepsilon^3 P_1(\xi,\,\tau) = \varepsilon^3\alpha\eta_1(\varepsilon,\,\tau)$$
$$\xi = \varepsilon(x - t), \quad \tau = \varepsilon^4 t$$

If we take the characteristic time for which the wave process is described by the Burgers equation as unity, then, since $\varepsilon \sim 0.1$, for the process described by the Korteweg-de Vries equation the characteristic time will be of the order of 10 dimensionless units and for the process described by the nonlinear evolutionary equation (3.14) the characteristic time will be of the order of 100 dimensionless units.

4. EXACT SOLUTIONS OF NONLINEAR WAVE EQUATIONS

It has been obtained that, when a fluid flows through a viscoelastic tube, at various instants of time the fluid velocity divided by the quantity $5/4 + \alpha_1/\alpha$ obeys the following evolutionary equations

$$u_t + uu_x = \frac{\lambda}{2\alpha}u_{xx} \tag{4.1}$$

$$u_t + uu_x + \frac{\gamma - \beta}{2\alpha}u_{xxx} = 0 \tag{4.2}$$

$$u_t + uu_x + \frac{\delta}{2\alpha}u_{xxxx} = 0 \tag{4.3}$$

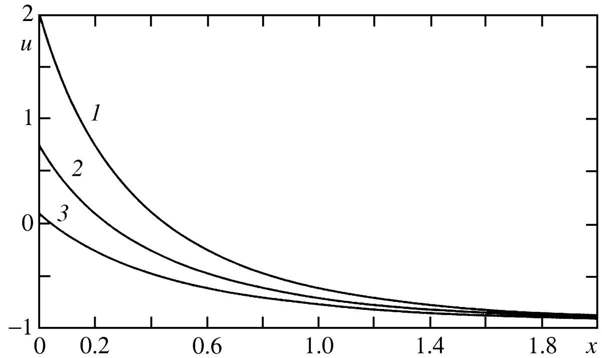

Fig. 1. Solution of Eq. (4.3) at instants $t = 0$, 0.2, and 0.4 (curves *1–3*, respectively)

The perturbations of the pressure and tube radius obey analogous equations.

The Burgers equation (4.1) and the Korteweg-de Vries equation (4.2) have been well studied. Using the Cole-Hopf transform [21, 22], equation (4.1) can be reduced to the linear heat-conduction equation

$$u = -\frac{\lambda}{\alpha}\frac{\partial \ln Z}{\partial x}, \qquad Z_t = \frac{\lambda}{2\alpha} Z_{xx}$$

For Eq. (4.2) the solution of the Cauchy problem can be found using the inverse scattering transform [23, 24].

We will study possible solutions of Eq. (4.3). In the traveling-wave variables $u(x,\ t) = y(z)$, where $z = x - C_0 t$, after integration over z, equation (4.3) takes the form:

$$C_1 - C_0 y + \frac{1}{2}y^2 + \nu y_{zzz} = 0, \qquad \nu = \frac{\delta}{2\alpha} \tag{4.4}$$

As distinct from Eqs. (4.1) and (4.2), equation (4.3) does not belong to the class of equations which can be solved exactly. This can be shown by checking the equations for the Painlevé property [24]. Assigning $y = a_0/(z - z_0)^p$, we obtain $p = 3$ and $a_0 = 120\nu$. Assigning $y = a_0/(z - z_0)^p + \Lambda(z - z_0)^{j-3}$ and equating the terms of the first degree in Λ to zero, we find the Fuchs indices for (4.4): $j_1 = -1$ and $j_{2,3} = (13 \pm i\sqrt{71})/2$.

The two Fuchs indices are complex conjugate; therefore, equation (4.4) and, consequently, the nonlinear evolutionary equation (4.3) do not belong to the class of equations which can be solved exactly.

In order to seek exact solutions of Eq. (4.4) we can use various methods. However, in what follows, we will use the simplest-equation method [25], recently proposed by one of the authors of the present study, which is a generalization of the approaches proposed earlier [26, 27].

As the general solution of Eq. (4.4) has a third-order pole, we can seek the solution of this equation in the form:

$$y(z) = A_0 + A_1 Y(z) + A_2 Y(z)^2 + A_3 Y(z)^3 \tag{4.5}$$

where it is assumed that $Y(z)$ must satisfy the Ricatti equation

$$Y_z = -Y^2 + aY(z) + b \tag{4.6}$$

After substituting (4.5) and (4.6) in Eq. (4.4) we find the coefficients

$$A_3 = 120\nu, \quad A_2 = -180\nu a, \quad A_1 = 90\nu a^2, \quad A_0 = C_0 + 15\nu a^3, \quad b = -a^2/4$$

When $C_1 = C_0^2/2$ the solution of Eq. (4.4) takes the form:

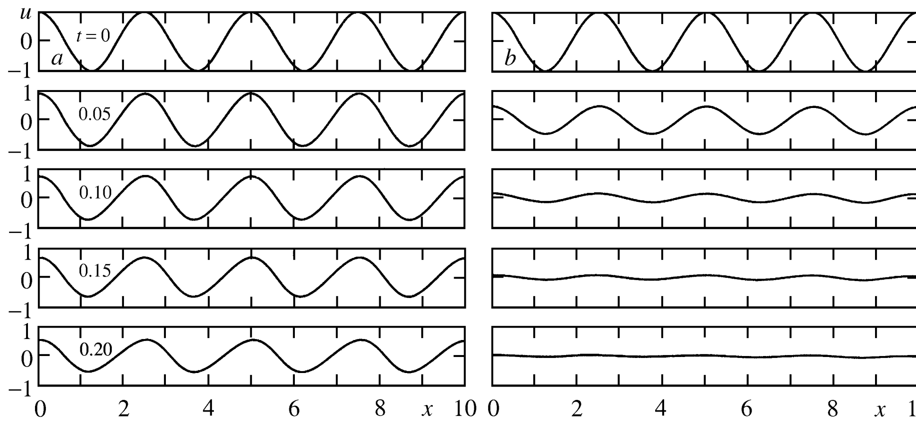

Fig. 2. Evolution of a periodic pressure wave described by the Burgers equation (*a*) and Eq. (4.3) (*b*)

$$u(x,\ t) = C_0 + \frac{120\nu}{(x - C_0 t + C_2)^3} \tag{4.7}$$

where C_0 and C_2 are arbitrary constants.

Figure 1 illustrates the solution of the problem described by Eq. (4.3) on a semi-infinite straight line for a given boundary condition at the point $x = 0$ corresponding to (4.7) at times $t_1 = 0$, $t_2 = 0.2$, and $t_3 = 0.4$. In constructing the solutions we used the following parameters: $C_0 = -1.0$, $\nu = 0.025$, and $C_2 = 1.0$. The exact solution (4.7) was used for testing the numerical solutions described by the nonlinear wave equation (4.3).

Periodic solutions of Eq. (4.4) can also be found by means of the simplest-equation method with allowance for the third-order pole of the general solution of Eq. (4.4). If as the simplest equation we take the equation for the Jacobi elliptic function, then the solution of (4.4) can be sought in the form:

$$y(z) = A_0 + A_1 Q + A_2 Q^2 + A_3 Q^3 + B_1 Q_z + B_2 Q Q_z \tag{4.8}$$

where the coefficients A_0, A_1, A_2, B_1, and B_2 can be found after substituting (4.8) in Eq. (4.4) and $Q(z)$ is a solution of the equation for the Jacobi elliptic function

$$Q_z^2 - Q^4 - aQ^3 - bQ^2 - cQ - d = 0 \tag{4.9}$$

Substituting (4.8) in Eq. (4.4) and taking into account Eq. (4.9), after equating the expressions with the same powers of $Q(z)$ to zero, we obtain

$$\begin{gathered} A_0 = C_0 \pm 15\nu c, \quad A_1 = \pm 30\nu b, \quad A_2 = \pm 45\nu a, \quad A_3 = \pm 60\nu \\ B_1 = -15\nu a, \qquad B_2 = -60\nu, \qquad d = \frac{1}{4}ac - \frac{1}{12}b^2 \\ C_1 = \frac{1}{2}C_0^2 - 20\nu^2 b^3 + \frac{135}{2}\nu^2 bca - \frac{135}{2}\nu^2 c^2 + \frac{45}{8}\nu^2 a^2 b^2 - \frac{135}{8}\nu^2 a^3 c \end{gathered} \tag{4.10}$$

The solution expressed in terms of Jacobi elliptic functions has the form:

$$y(z) = C_0 \pm 15\nu c \pm 30 b\nu Q \pm 45 a\nu Q^2 \pm 60\nu Q^3 - 15 a\nu Q_z - 60\nu Q Q_z$$

Here, $Q(z)$ must satisfy Eq. (4.9) subject to limitation (4.10) on the parameter d.

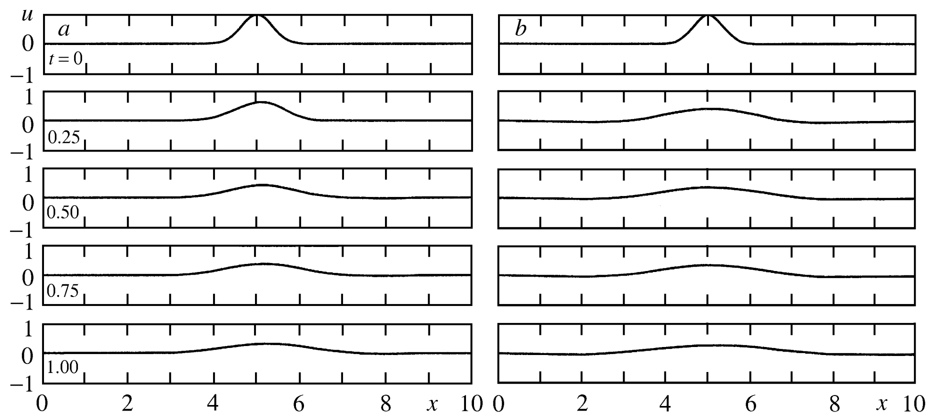

Fig. 3. Evolution of a solitary pressure wave as described by the Burgers equation (4.1) (*a*) and by Eq. (4.3) (*b*)

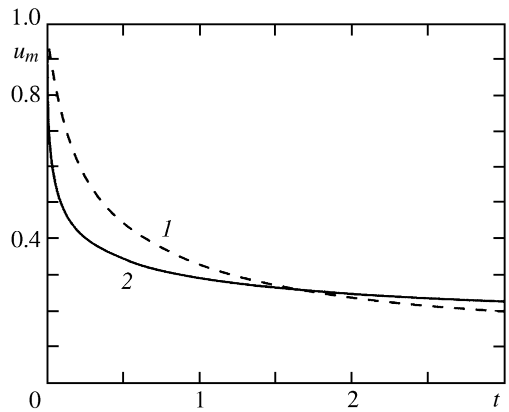

Fig. 4. Time dependence of the solitary pressure wave amplitude u_m as described by the Burgers equation (curve *1*) and Eq. (4.3) (curve *2*)

5. NUMERICAL SOLUTIONS OF PROBLEMS OF PROPAGATION OF PERTURBATIONS THROUGH A VISCOELASTIC TUBE

At present, there are no difficulties in numerically simulating the transfer through a viscoelastic tube of a pressure pulse described by the Burgers and Korteweg-de Vries equations. The evolution of the pulse dynamics described by these equations has been studied in detail (see, for example, [28–30]).

With time a pulse described by the Burgers equation decreases in amplitude due to dissipation and its shape is simultaneously distorted due to nonlinearity.

The Korteweg-de Vries equation has solutions in the form of solitons transported in the medium without distortion or dissipation. If the initial pulse does not correspond to the soliton solution, the original pulse may break down into several solitons, each transported without distortion and at a constant velocity which depends on the amplitude.

We will now consider the pressure pulse propagation through a viscoelastic tube described by Eq. (4.3). For the numerical calculations we used an implicit two-layer five-point finite-difference scheme. For the exact solution (4.7) the scheme gives a relative error of less than one per cent for coordinate and time steps $h = 0.05$ and $\tau = h^2$.

Comparing the pressure pulses described by the Burgers equation and Eq. (4.3) for periodic boundary conditions, it is possible to draw conclusions concerning the features of the waves obeying these equations.

As the initial profile, we took a periodic pressure pulse in the form $u(x,\ 0) = \cos(4/5\pi x)$ and a solitary Gaussian pressure pulse $u(x,\ 0) = \exp(-4(x - 5)^2)$ on a segment of length $l = 10$. In the calculations we assumed that α, δ, and $\lambda = 1$.

The higher derivatives in Eqs. (4.1) and (4.3) describe the dissipative processes. However, the dissipation of the waves described by Eqs. (4.1) and (4.3) takes place differently (Figs. 2–4). For nonlinear waves of the fourth-order evolutionary equation (4.3) the pressure pulses are additionally smoothed. In the linearized case the shortwave harmonics in the initial profile are more rapidly damped than in the case of the Burgers equation (4.1), while the longwave harmonics are more slowly damped.

Summary. Among the features of nonlinear wave propagation through a viscoelastic tube, we note that in the initial stage (characteristic time t of the order of ε^{-1}) the perturbations are damped in accordance with the Burgers equation (4.1). In this stage the main wave dissipation factor is the resistance of the medium in to the motion of the wall. In the second stage ($t \sim \varepsilon^{-2}$) the wave propagation can be described by the Korteweg-de Vries equation (4.2). In this case the nonlinear waves travel without distortion of the wave shape. The determining factor here is the purely elastic properties of the wall. In the third stage ($t \sim \varepsilon^{-3}$) the nonlinear wave propagation obeys Eq. (4.3). This stage is characterized by smearing (damping) of the waves. In this case the viscous properties of the wall are the main factor. The steepness of the wave profile is determined by the nonlinear wall elasticity over the entire evolution of the perturbation.

The authors wish to thank A. V. Migita for useful discussions and the referee for valuable comments. The work was carried out with financial support from the International Scientific and Technological Center (project V-1213).

REFERENCES

1. M. O'Rourke, "Mechanical principles in arterial disease," *Hypertension*, **26**, No. 1, 2–9 (1995).
2. G. London, "Remodelling of arteries and the blood pressure of uraemia patients," *Nefrologiya i dializ*, **2**, No. 3, 11 (2000).
3. S. N. Polivoda and A. A. Cherepok, "Remodelling of arterial vessels for hypertensive patients," *Ukr. Zh. Kardiologii*, No. 6, 9 (2003).
4. Y. C. Fung, *Biomechanics: Mechanical Properties of Living Tissues*, Springer, N. Y. etc. (1993).
5. R. H. Cox, "Comparison of linearized wave propagation models for arterial blood flow analysis," J. Biomechanics, **2**, No. 3, 251–265 (1969).
6. C. G. Caro, T. J. Pedly, R. C. Schroter, W. A. Seed, *The Mechanics of the Circulation*, Oxford University Press, Oxford etc. (1978).
7. S. A. Regirer, "Some problems of hydrodynamics of the blood circulation," in: *Hydrodynamics of the Blood Circulation. Appendix* [in Russian], Moscow, Mir (1971), pp. 242–264.
8. S. Miekisz, "Non-linear theory of viscous flow in elastic tubes," *Phys. Med. Biol.*, **6**, No. 1, 103–109 (1961).
9. S. A. Regirer and N. Kh. Shadrina, "Elementary model of a vessel with a wall sensitive to mechanical stimulus," *Biofizika*, **47**, No. 5, 908–913 (2002).
10. S. Čanic and A. Mikeli, "Effective equations modeling the flow of a viscous incompressible fluid through a long elastic tube arising in the study of blood flow through small arteries," *SIAM J. Appl. Dynam. Systems*, **2**, No. 3, 431–463 (2003).
11. L. Formaggia, D. Lamponi, A. Quarteroni, "One-dimensional models for blood flow in arteries," *J. Eng. Math.*, **47**, No. 3–4, 251–276 (2003).
12. H. Demiray, "On some nonlinear waves in fluid-filled viscoelastic tubes: weakly dispersive case," *Communs. Nonlinear Sci. and Numer. Simulation*, **10**, No. 4, 425–440 (2005).
13. G. Pontrelli, "A multiscale approach for modelling wave propagation in an arterial segment," *Comput. Methods in Biomech. and Biomed. Eng.*, **7**, No. 2, 79–89 (2004).
14. R. C. Cascaval, "Variable coefficient KdV equations and waves in elastic tubes," in: *Lecture Notes in Pure and Appl. Math. Evolution Equations*, **234** (2003).
15. A. N. Volobuev, "Fluid flow through tubes with elastic walls," *Uspekhi Fiz. Nauk*, **165**, No. 2, 177–186 (1995).
16. A. Quarteroni, M. Tuveri, and A. Veneziani, "Computational vascular fluid dynamics: problems, models and methods," *Comp. Vis. Science*, **2**, No. 4, 163–197 (2000).
17. L. D. Landau and E. M. Lifshitz, *Theoretical Physics. Vol. 7. Elasticity Theory* [in Russian], Fizmatlit, Moscow (2001).

18. J. T. Ottesen, "Valveless pumping in a fluid-filled closed elastic tube-system: one-dimensional theory with experimental validation," *J. Math. Biol.*, **46**, No. 4, 309–332 (2003).
19. C. H. Su and C. S. Gardner, "Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries and the Burgers equations," *J. Math. Phys.*, **10**, No. 5, 536–539 (1969).
20. E. Witzleb, "Functions of the vascular system," in: R. F. Schmidt and G. Thews (Eds.), *Human Physiology*, Vol. 3, Springer, Berlin (1983).
21. E. Hopf, "The partial differential equation $u_t + uu_x = u_{xx}$," *Communs. Pure Appl. Math.*, **3**, No. 3, 201–230 (1950).
22. J. D. Cole, "On a quasi-linear parabolic equation occurring in aerodynamics," *Quart. Appl. Math.*, **9**, No. 3, 225–236 (1950).
23. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Method for solving the Korteweg-de Vries equation," *Phys. Rev. Lett.*, **19**, No. 19, 1095–1097 (1967).
24. N. A. Kudryashov, *Analytic Theory of Nonlinear Differential Equations* [in Russian], Institute of Computer-Based Research, Moscow-Izhevsk (2004).
25. N. A. Kudryashov, "Simplest equation method to look for exact solutions of nonlinear differential equations," *Chaos, Solitons and Fractals*, **24**, No. 5, 1217–1231 (2005).
26. N. A. Kudryashov, "Exact soliton-like solutions of a generalized evolutionary equation in wave dynamics," *Prikl. Mat. Mekh.*, **52**, No. 3 465–470 (1988).
27. N. A. Kudryashov, "Exact solutions of the generalized Kuramoto-Sivashinsky equation," *Phys. Lett. A.*, **147**, No. 5/6, 287–291 (1990).
28. P. S. Landa, *Nonlinear Vibrations and Waves* [in Russian], Nauka. Fizmatlit, Moscow (1997).
29. A. C. Newell, *Solitons in Mathematics and Physics*, Society for Industrial and Applied Mathematics, Philadelphia (1985).
30. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Studies in Applied Mathematics, Vol. 4, Society for Industrial and Applied Mathematics, Philadelphia (1981).

E-mail: kudryashov@mephi.ru