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Solving the Polynomial Eigenvalue Problem by Linearization

Nick Higham
School of Mathematics
The University of Manchester

higham@ma.man.ac.uk
http://www.ma.man.ac.uk/~higham/

Joint work with Ren-Cang Li, Steve Mackey and Françoise Tisseur.

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Outline

PEP and Linearization Background

B'err and Conditioning of Linearizations

Algorithm based on Linearization

Polynomial Eigenproblem (PEP)

$$P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i, \qquad A_i \in \mathbb{C}^{n\times n}, \quad A_m \neq 0.$$

P assumed **regular** ($\det P(\lambda) \not\equiv 0$).

Find scalars λ and nonzero vectors x and y satisfying $P(\lambda)x = 0$ and $y^* P(\lambda) = 0$.

Special case quadratic eigenvalue problem (QEP):

$$(\lambda^2 M + \lambda D + K)x = 0.$$

Applications

- classical structural mechanics
- molecular dynamics
- gyroscopic systems
- optical waveguide design
- MIMO systems in control theory
- constrained least squares problems.

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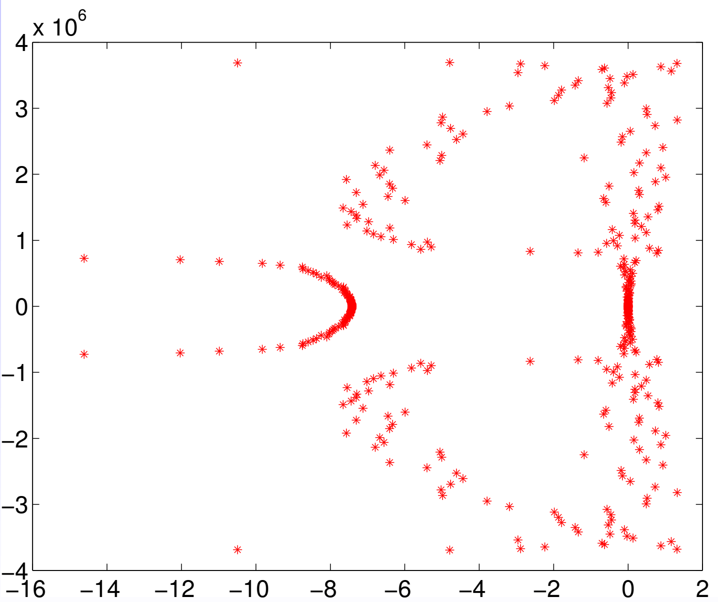
- constrained least squares problems.

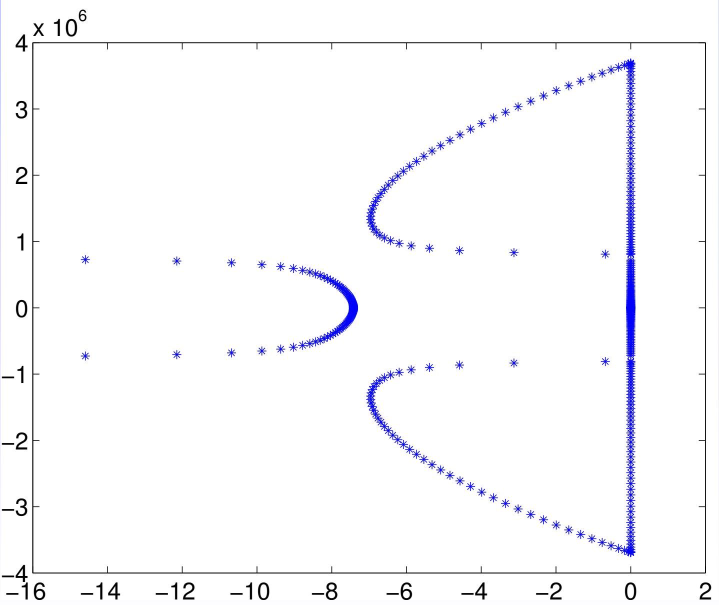
More specifically:

- Vibration of rail tracks by high speed trains (SIAM News, Nov. 2004).
- Extreme designs lead to problems with poor conditioning; physics of system leads to structure.

Polyeig (Companion Pencil)



Polyeig on Scaled Quadratic



Methods

Interested in methods for solving dense problems (possibly a projection of a sparse problem).

- Solvent: $\sum_{i=0}^{m} A_i X^i = 0$.
- Bandwidth reduction.
- Structure-preserving transformations.
- **Linearization**.

Current Software

- MATLAB's `polyeig`.
- Solvers in commercial engineering packages (Nastran, etc.).

Aim to design and implement LAPACK solvers for *general* and *structured* QEPs/PEPs (Manchester and TU Berlin).

Linearizations

$$L(\lambda) = \lambda X + Y, \qquad X, Y \in \mathbb{C}^{mn \times mn}$$

is a **linearization** of $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ if

$$E(\lambda) L(\lambda) F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

for some **unimodular** $E(\lambda)$ and $F(\lambda)$.

Example

Companion form linearization

$$E(\lambda) \left(\lambda \begin{bmatrix} A_2 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ -I & 0 \end{bmatrix} \right) F(\lambda) = \begin{bmatrix} \lambda^2 A_2 + \lambda A_1 + A_0 & 0 \\ 0 & I \end{bmatrix}.$$

Solution Process for PEP

- **Linearize** $P(\lambda)$ into $L(\lambda) = \lambda X + Y$.
- Solve **generalized eigenproblem** $L(\lambda)z = 0$.
- **Recover** eigenvectors of P from those of L.

Usual choice of L: companion linearization, for which

$$z = \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ \lambda x \\ x \end{bmatrix}.$$

Left e'vec: more complicated formula.

Desiderata for a Linearization

- Good conditioning.
- Backward stability.
- Suitable **eigenvector recovery** formulae.
- Preservation of structure, e.g. **symmetry**.
- Numerical preservation of key **qualitative** properties, including location and symmetries of spectrum.
- Preserve partial multiplicities of e'vals (strong linearization).

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Backward Error

$$P(\alpha,\beta) = \sum_{i=0}^{m} \alpha^i \beta^{m-i} A_i \qquad (\lambda \equiv \alpha/\beta).$$

(x,α,β) approx eigenpair of P:

$$\eta_P(x,\alpha,\beta) = \min\Big\{ \epsilon : \sum_{i=0}^{m} \alpha^i \beta^{m-i}(A_i + \Delta A_i)x = 0, \\ \|\Delta A_i\|_2 \le \epsilon \|A_i\|_2,\ i = 0\colon m \Big\}.$$

Extending Tisseur (2000):

$$\eta_P(x,\alpha,\beta) = \frac{\|P(\alpha,\beta)x\|_2}{\left(\sum_{i=0}^{m} |\alpha|^i |\beta|^{m-i} \|A_i\|_2\right) \|x\|_2}.$$

Key Question

★ How **good** an **approx eigenpair** of P will be produced from an approx eigenpair of L?
Here "good" refers to

- relative error (see H, D. S. Mackey & Tisseur, 2005),
- backward error.

A small perturbation to

$$C_1(\lambda) = \begin{bmatrix} A_2 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ I & 0 \end{bmatrix}$$

may not correspond to a small perturbation to $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$.

Want to compare

$$\eta_P(x,\alpha,\beta) = \frac{\|P(\alpha,\beta)x\|_2}{\left(\sum_{i=0}^m |\alpha|^i|\beta|^{m-i}\|A_i\|_2\right)\|x\|_2},$$

with

$$\eta_L(z,\alpha,\beta) = \frac{\|L(\alpha,\beta)z\|_2}{(|\alpha|\|X\|_2 + |\beta|\|Y\|_2)\|z\|_2}.$$

(z,α,β): **approx** e'pair of linearization $L(\lambda) = \lambda X + Y$ of $P(\lambda)$.
(x,α,β): **approx** e'pair of P with x **recovered from** z.

One Sided Factorization

Suppose there exists $n \times nm$ $G(\alpha, \beta)$ s.t.

$$G(\alpha,\beta)L(\alpha,\beta) = g^T \otimes P(\alpha,\beta), \quad g \in \mathbb{C}^m.$$

With $z_i := z((i-1)n+1: in)$,

$$\begin{aligned} G(\alpha,\beta)L(\alpha,\beta)z &= (g^T \otimes P(\alpha,\beta))z \\ &= [g_1 P(\alpha,\beta) \ \dots \ g_m P(\alpha,\beta)] \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \\ &= P(\alpha,\beta) \sum_{i=1}^m g_i z_i =: P(\alpha,\beta)x. \end{aligned}$$

$$\Rightarrow \ \|P(\alpha,\beta)x\|_2 \le \|G(\alpha,\beta)\|_2 \|L(\alpha,\beta)z\|_2.$$

Bounding η_P/η_L

Suppose $G(\alpha,\beta)L(\alpha,\beta) = g^T \otimes P(\alpha,\beta)$, $g \in \mathbb{C}^m$.

Let z be approx e'vec of L with e'val (α,β) and take $x = \sum g_i z_i$ as approx e'vec of P.

$$\frac{\eta_P(x,\alpha,\beta)}{\eta_L(z,\alpha,\beta)} = \frac{|\alpha|\,\|X\|_2 + |\beta|\,\|Y\|_2}{\sum_{i=0}^m |\alpha|^i|\beta|^{m-i}\|A_i\|_2} \cdot \frac{\|P(\alpha,\beta)x\|_2}{\|L(\alpha,\beta)z\|_2} \cdot \frac{\|z\|_2}{\|x\|_2}.$$

Bounding η_P/η_L

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- **Separates the dependence** on L, P and (α,β) from dependence on G and z.
- η_P is **finite** if η_L is.

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- **Separates the dependence** on L, P and (α,β) from dependence on G and z.
- η_P is **finite** if η_L is.
- Sim. for left e'vecs: assume

$$L(\alpha,\beta)H(\alpha,\beta) = h \otimes P(\alpha,\beta), \quad h \in \mathbb{C}^m.$$

Vector Spaces $\mathbb{L}_1$, $\mathbb{L}_2$, and $\mathbb{DL}(P)$

$$\Lambda := [\lambda^{m-1}, \lambda^{m-2}, \ldots, 1]^T.$$

Mackey, Mackey, Mehl & Mehrmann (2005) define

$$\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m \,\},$$
$$\mathbb{L}_2(P) = \{\, L(\lambda) : (\Lambda^T \otimes I_n)L(\lambda) = \widetilde{v}^T \otimes P(\lambda),\ \widetilde{v} \in \mathbb{C}^m \,\},$$
$$\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P).$$

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- Dimensions: $\mathbb{L}_1$, $\mathbb{L}_2$: $m(m-1)n^2 + m$, $\mathbb{DL}$: m.
- Pencils in $\mathbb{DL}(P)$ are block symmetric.
- Almost all L in the above are linearizations.

Companion Linearizations

$C_i(\lambda) = \lambda X + Y_i$ with $X = \mathrm{diag}(A_m, I_n, \dots, I_n)$,

$$Y_1 = \begin{bmatrix} A_{m-1} & A_{m-2} & \dots & A_0 \\ -I_n & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -I_n & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} A_{m-1} & -I_n & \dots & 0 \\ A_{m-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \dots & 0 \end{bmatrix}.$$

- $C_1 \in \mathbb{L}_1(P)$ ($v = e_1$) and $C_2 \in \mathbb{L}_2(P)$ ($\widetilde{v} = e_1$).
- C_1 and C_2 *always* (strong) linearizations.
- $C_2(P) = C_1(P^T)^T \Rightarrow$ concentrate on C_1 only.

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Are the factorizations

$$G(\alpha,\beta)C_1(\alpha,\beta) = g^T \otimes P(\alpha,\beta),$$
$$C_1(\alpha,\beta)H(\alpha,\beta) = h \otimes P(\alpha,\beta)$$

possible for C_1?

First Companion

- For $m = 2$,

$$\begin{bmatrix} \alpha I & -\beta A_0 \\ \beta I & \beta A_1 + \alpha A_2 \end{bmatrix} C_1(\alpha,\beta) = \begin{bmatrix} P(\alpha,\beta) & 0 \\ 0 & P(\alpha,\beta) \end{bmatrix}$$

  yields two choices:

$$\begin{aligned} G(\alpha,\beta) &= [\,\alpha I \quad -\beta A_0\,], & g = e_1 \Rightarrow x = z_1, \\ G(\alpha,\beta) &= [\,\beta I \quad \beta A_1 + \alpha A_2\,], & g = e_2 \Rightarrow x = z_2. \end{aligned}$$

  Generalizes to arbitrary degrees m.

- $C_1 \in \mathbb{L}_1$ $(v = e_1) \Rightarrow C_1(\alpha,\beta)(\Lambda_{\alpha,\beta} \otimes I_n) = e_1 \otimes P(\alpha,\beta)$.
  Can take $H(\alpha,\beta) = \Lambda_{\alpha,\beta} \otimes I_n$ and $h = e_1 \Rightarrow y = w_1$.

First Companion: Right Eigenvector

Theorem

Let z be approx right e'vec of C_1 with approx e'val (α, β).
For $z_k = z((k-1)n+1: kn)$, $k = 1: m$,

$$\frac{\eta_P(z_k, \alpha, \beta)}{\eta_{C_1}(z, \alpha, \beta)} \le m^{5/2} \frac{\max\bigl(1, \max_i \|A_i\|_2\bigr)^2}{\min\bigl(\|A_0\|_2, \|A_m\|_2\bigr)} \frac{\|z\|_2}{\|z_k\|_2}.$$

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$\eta_P \approx \eta_{C_1}$ if

- $\min(\|A_0\|_2, \|A_m\|_2) \approx \max_i \|A_i\|_2 \approx 1$.
- $\|z\|_2 / \|z_k\|_2 \approx 1$.

Exact $z = \Lambda_{\alpha,\beta} \otimes x \Rightarrow \dfrac{\|z\|_2}{\|z_k\|_2} \le \sqrt{m}, \quad k = \begin{cases} 1 \text{ if } |\alpha| \ge |\beta|, \\ m \text{ if } |\alpha| \le |\beta|. \end{cases}$

First Companion: Left Eigenvector

Theorem

Let w be approx left e'vec of C_1 with approx e'val (α, β). Then for $w_1 = w(1:n)$,

$$\frac{\eta_P(w_1{}^*, \alpha, \beta)}{\eta_{C_1}(w^*, \alpha, \beta)} \le m^{3/2} \frac{\max(1, \max_i \|A_i\|_2)}{\min(\|A_m\|_2, \|A_0\|_2)} \frac{\|w\|_2}{\|w_1\|_2}.$$

$\eta_P(w_1{}^*, \alpha, \beta) \approx \eta_{C_1}(w^*, \alpha, \beta)$ if

- $\min(\|A_0\|_2, \|A_m\|_2) \approx \max_i \|A_i\|_2 \approx 1$
- $\|w\|_2 / \|w_1\|_2 \approx 1$.

Left Eigenvector recovery for C_1

Lemma

$y \in \mathbb{C}^n$ is a left e'vec of P with simple e'val (α, β) iff

$$w = \begin{bmatrix} [\alpha^{m-1} I]^* \\ -[\alpha^{m-2}\beta A_{m-2} + \cdots + \alpha\beta^{m-2} A_1 + \beta^{m-1} A_0]^* \\ -[\alpha^{m-2}\beta A_{m-3} + \cdots + \alpha^2\beta^{m-3} A_1 + \alpha\beta^{m-2} A_0]^* \\ \vdots \\ -[\alpha^{m-2}\beta A_0]^* \end{bmatrix} y, \quad \alpha \neq 0,$$

is a left e'vec of C_1 corr. to (α, β). Every left e'vec of C_1 with e'val (α, β) has this form for some left e'vec y of P.

Left Eigenvector recovery for C_1

Lemma

$y \in \mathbb{C}^n$ is a left e'vec of P with simple e'val (α, β) iff

$$w = \begin{bmatrix} [\beta^{m-1} I]^* \\ [\alpha\beta^{m-2} A_m + \beta^{m-1} A_{m-1}]^* \\ \vdots \\ [\alpha^{m-1} A_m + \cdots + \alpha\beta^{m-2} A_2 + \beta^{m-1} A_1]^* \end{bmatrix} y, \quad \alpha = 0$$

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is a left e'vec of C_1 corr. to (α, β). Every left e'vec of C_1 with e'val (α, β) has this form for some left e'vec y of P.

If $\|A_i\|_2 \approx 1\ \forall i$ then exact left e'vec w satisfies $\|w\|_2 / \|w_1\|_2 \le (m^2/3)^{1/2}$.

UnScaled Companion Form

$C_1(\lambda) = \lambda X + Y_i$ with $X = \mathrm{diag}(A_m, I_n, \ldots, I_n)$,

$$Y_1 = \begin{bmatrix} A_{m-1} & A_{m-2} & \ldots & A_0 \\ -I_n & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & -I_n & 0 \end{bmatrix}.$$

Scaled Companion Form

$C_1(\lambda) = \lambda X + Y_i$ with $X = \mathrm{diag}(A_m, \mu I_n, \dots, \mu I_n)$,

$$Y_1 = \begin{bmatrix} A_{m-1} & A_{m-2} & \dots & A_0 \\ -\mu I_n & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -\mu I_n & 0 \end{bmatrix}.$$

Let $D_\mu = \mathrm{diag}(1, \mu, \dots, \mu) \otimes I_n$, where $\mu = \max_i \|A_i\|_2$.
Then $D_\mu C_1(\lambda) \in \mathbb{L}_1(P)$ with $v = e_1$.

Scaled Companion: Right and Left E'vecs

$$\rho = \frac{\max_i \|A_i\|_2}{\min(\|A_0\|_2, \|A_m\|_2)}.$$

Theorem

Let z and w be approx right and left e'vecs of $D_\mu C_1$ corr. to the approx e'val (α, β). Then

$$\frac{\eta_P(z_k, \alpha, \beta)}{\eta_{D_\mu C_1}(z, \alpha, \beta)} \le m^{5/2}\,\rho \frac{\|z\|_2}{\|z_k\|_2}, \quad k = 1\colon m,$$

$$\frac{\eta_P(w_1^*, \alpha, \beta)}{\eta_{D_\mu C_1}(w^*, \alpha, \beta)} \le m^{3/2}\,\rho \frac{\|w\|_2}{\|w_1\|_2}.$$

Comments

- Square has gone from right e'vec bound.
- Bounds now scale-independent ($A_i \leftarrow \theta A_i$).
- $\|w\|_2/\|w_1\|_2 \approx 1$ now guaranteed for all $\|A_i\|_2$.
- More sophisticated scalings, including balancing, may destroy $\mathbb{L}_1$ property.

Scaling $Q(\lambda)$

Write $a = \|A\|_2$, $b = \|B\|_2$, $c = \|C\|_2$.

Fan, Lin & Van Dooren (2004): let $\lambda = \mu\gamma$,

$$Q(\lambda) = \lambda^2 A + \lambda B + C \to \widetilde{Q}(\mu) = \mu^2(\delta\gamma^2 A) + \mu(\delta\gamma B) + \delta C,$$

where

$$\gamma = \sqrt{c/a}, \quad \delta = 2/(c + b\gamma).$$

For $\widetilde{Q}(\mu) = \mu^2\widetilde{A} + \mu\widetilde{B} + \widetilde{C}$ we have

$$\max(\|\widetilde{A}\|_2, \|\widetilde{B}\|_2, \|\widetilde{C}\|_2) \le 2.$$

Growth Factor Bound

$$\frac{\eta_{\widetilde{Q}}(z_i,\alpha,\beta)}{\eta_{C_1}(z,\alpha,\beta)} \le 2^{7/2}\omega\,\frac{\|z\|_2}{\|z_i\|_2}, \qquad i = 1, 2,$$

where, with $|\alpha|^2 + |\beta|^2 = 1$,

$$1 \le \omega \le \min\left\{1+\tau, \frac{1}{|\alpha\beta|}\right\} \le 1+\tau, \qquad \tau = \frac{b}{\sqrt{ac}}.$$

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$$1 \le \omega \le \min\left\{1+\tau, \frac{1}{|\alpha\beta|}\right\} \le 1+\tau, \qquad \tau = \frac{b}{\sqrt{ac}}.$$

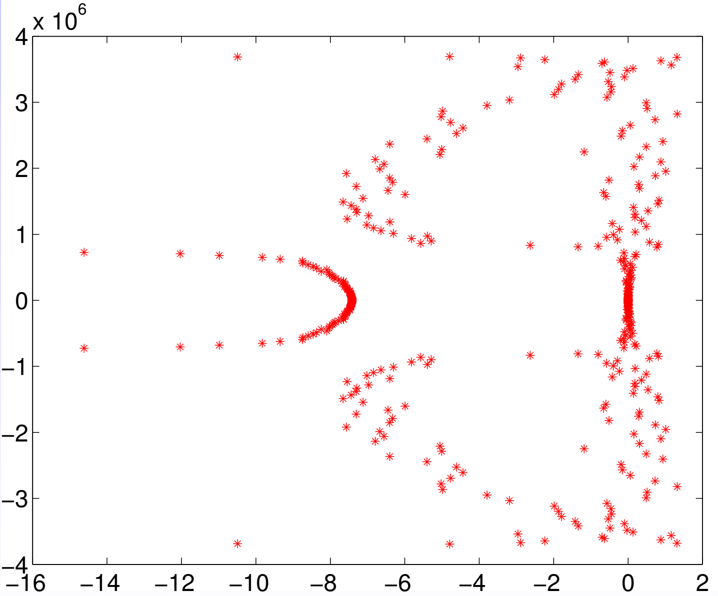
- F, L & VD identify $\max(1+\tau, 1+\tau^{-1})$ as growth factor.
- Our bounds for ω sharper:
  - $\tau \ll 1$ is harmless.
  - if $\tau \gg 1$, $\min\{\cdot\} = O(1)$ if $|\alpha||\beta| = O(1)$.
- $\omega = O(1)$ if $\|B\|_2 \lesssim \sqrt{\|A\|_2\|C\|_2}$. Hence $\eta_L \approx \eta_P$ for systems not heavily damped.

Comparison with Conditioning Results

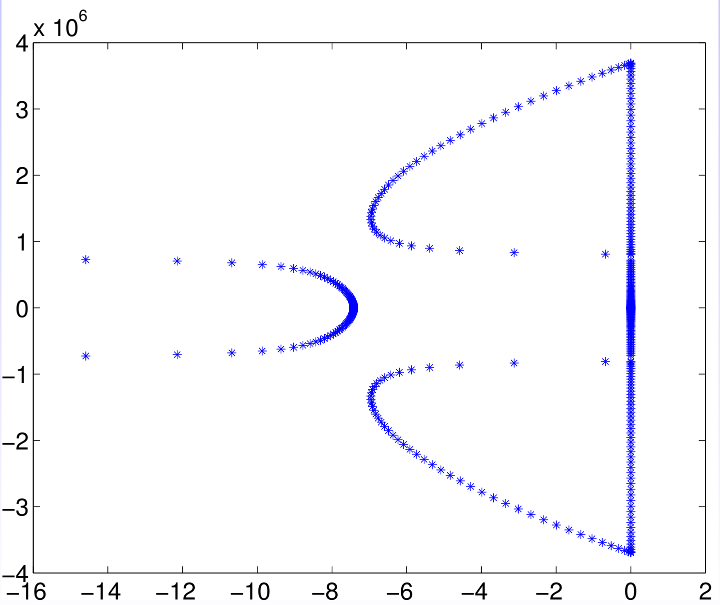
- Analysis of H, D. S. Mackey & Tisseur (2005) bounds the ratio $\kappa_{C_1}(\alpha, \beta)/\kappa_P(\alpha, \beta)$.
- Conditions for those $\kappa_{C_1}(\alpha, \beta) \approx \kappa_P(\alpha, \beta)$ are *"essentially the same"* as those for $\eta_{C_1} \approx \eta_P$.

Backward error results entirely harmonious with e'val conditioning results.

Polyeig (Companion Pencil)



Polyeig on Scaled Quadratic



Details

	Before scaling	After scaling
$\|A\|_2$	10^{-2}	10^{0}
$\|B\|_2$	10^{0}	10^{-3}
$\|C\|_2$	10^{9}	10^{0}
	$\rho = 10^{11}$	$\rho = 1$, $\omega = 1$

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Algorithm based on Linearization

Meta-Algorithm for PEP

```
Preprocess P
for one or more (scaled) linearizations L
    Balance L
    Apply QZ to L (maybe HZ if structured)
    Obtain relevant e'vals
    Recover left and right e'vecs
    Iteratively refine e'vecs
    Compute/estimate b'errs and condition numbers
    Detect nonregular problem
end
```

Balancing

Balancing GEP:

- Ward (1981)
- Lemonnier & Van Dooren (2005)

To investigate:

- Exploit structure of pencils arising via linearization of a matrix poly.
- Can we balance a QEP?
- To what extent can balancing make the results worse? Cf. Watkins (2006): *A Case where Balancing is Harmful*.

Iterative Refinement

- Underlying theory for fixed and extended precision residuals in Tisseur (2001).
- Done for definite GEPs in Davies, H & Tisseur (2001).
- Details for QEPs in Berhanu (2005), incl. complex conj. pairs in real arith.

Issues:

- Convergence to wrong eigenpair or non-convergence.
- Exploiting structure of pencil from a linearization.

Tentative Outline of Algorithm for QEP

1. Preprocess Q: Fan, Lin & Van Dooren (2004) scaling
2. Let $L = D_\mu C_1$: scaled companion linearization
3. Apply QZ to L
4. Obtain relevant e'vals
5. Recover left (w_1) and right ($\max_i \|z_i\|_2$) e'vecs
6. Compute/estimate b'errs and condition numbers
7. Detect nonregular problem

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Main `polyeig` differences:

- Doesn't scale.
- Doesn't return left e'vecs.
- Doesn't detect nonregular Q.

Concluding Remarks

- ★ Analysis of cond. & b'err for wide variety of lineariz'ns.
- ★ E'vector recovery formulae *crucial*.
- ★ Scaling *crucial*.
- ★ Favour $L =$ (scaled) companion form for general PEPs.
- ★ $L \in \mathbb{L}_1(P)$, $\mathbb{L}_2(P)$ or $\mathbb{DL}(P)$ for structured problems.
- ★ Poised to develop a general PEP algorithm & code.
- ★ Further work needed on
  - ★ algorithmics
  - ★ scaling & balancing (pencils & general m)
  - ★ structured problems (symm, odd–even, definite)
  - ★ ...

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