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Can You Count on Your Correlation Matrix?

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My Research Interests

Numerical analysis, numerical linear algebra.

Finance-related topics:

- **Correlation matrices**.
- **Matrix roots**, $A^{1/p}$.
  E.g., roots of transition matrices P in credit risk.

Questions From Finance Practitioners

"Given a real symmetric matrix A which is almost a correlation matrix what is the best approximating (in Frobenius norm?) correlation matrix?"

"I am researching ways to make our company's correlation matrix positive semi-definite."

"Currently, I am trying to implement some real options multivariate models in a simulation framework. Therefore, I estimate correlation matrices from inconsistent data set which eventually are non psd."

Correlation Matrix

An $n \times n$ symmetric positive semidefinite matrix A with $a_{ii} \equiv 1$.

Properties:

- symmetric,
- 1s on the diagonal,
- off-diagonal elements between -1 and 1.
- eigenvalues nonnegative.

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Is this a correlation matrix?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

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Spectrum: -0.4142, 1.0000, 2.4142.

Spectrum of Correlation Matrix

Theorem (Schur, Horn)

A necessary and sufficient condition for a symmetric $n \times n$ A to have e'vals $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and diagonal elements $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ (in any order along the diagonal) is that

$$\sum_{i=1}^{k} \lambda_i \le \sum_{i=1}^{k} \alpha_i, \qquad k = 1\colon n,$$

with equality for $k = n$.

Conclusion

For a correlation matrix any set of $\lambda_i \ge 0$ summing to n is possible.

Generating Random Correlation Matrices

- Efficient alg of **Bendel & Mickey** (1978) transforms a given symm pos semidef matrix with $\sum_i \lambda_i = n$ into a correlation matrix.
- Improved by **Davies & Higham** (2000).
- Implemented in NAG routine **`G05GBF`**.
- Useful for simulation purposes.

Stock Research

- Sample correlation matrices constructed from vectors of stock returns.
- Can compute sample correlations of pairs of stocks based on days on which both stocks have data available.
- Resulting matrix of correlations is **approximate**, since built from inconsistent data sets.
- Relatively few vectors of observations available, so approximate correlation matrix has **low rank**.

How to Proceed

- $\surd$ Plug the gaps in the missing data, then compute an exact correlation matrix.
- $\times$ Make ad hoc modifications to matrix: e.g., shift negative e'vals up to zero then diagonally scale.
- $\surd$ Compute the **nearest correlation matrix**.

Problem

Compute distance

$$\gamma(A) = \min\{ \|A - X\| : X \text{ is a correlation matrix} \}$$

and a matrix achieving the distance.

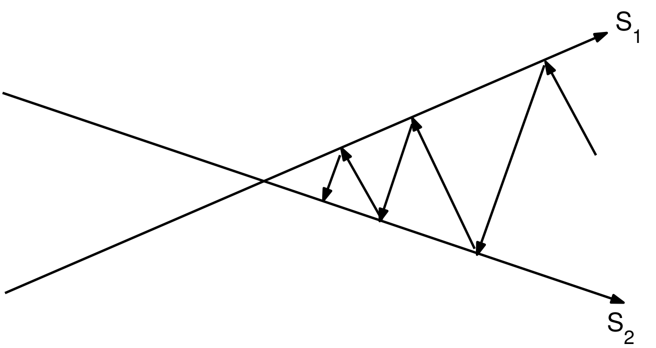
Use a weighted Frobenius norm:

- $\|A\|_W = \|W^{1/2} A W^{1/2}\|_F$ (W pos def),
- $\|A\|_H = \|H \circ A\|_F$ ($h_{ij} > 0$),

where $\|A\|_F^2 = \sum_{i,j} a_{ij}^2$.

Alternating Projections

von Neumann (1933), for subspaces.



Dykstra (1983) incorporated corrections for closed convex sets.

Projections

For $W \equiv I$.

- For $A = Q\,\mathrm{diag}(\lambda_i)Q^T$ let

$$\mathbf{P_S}(A) := Q\,\mathrm{diag}(\max(\lambda_i, 0))Q^T.$$

- $\mathbf{P_U}(A)$: replace diagonal by 1s.

More complicated for general W; see Higham (2002).

Algorithm (Higham, 2002)

Given symmetric $A \in \mathbb{R}^{n\times n}$ this algorithm computes nearest correlation matrix:

$\Delta S_0 = 0,\ Y_0 = A$
for $k = 1, 2, \ldots$
    $R_k = Y_{k-1} - \Delta S_{k-1}$ % Dykstra's correction.
    $X_k = \mathbf{P_S}(R_k)$
    $\Delta S_k = X_k - R_k$
    $Y_k = \mathbf{P_U}(X_k)$
end

- X_k and Y_k both converge to solution.
- $O(n^3)$ operations per step.
- Linear convergence.
- Can add further constraints/projections ...

Property of Iterates

Assume W is diagonal and $a_{ii} \geq 1$, $i = 1\colon n$.

Theorem

If A has t nonpositive e'vals then R_k has at least t nonpositive e'vals and X_k has at least t zero e'vals, for all k.

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If A has t nonpositive e'vals then R_k has at least t nonpositive e'vals and X_k has at least t zero e'vals, for all k.

- If t large or small can get $\mathbf{P_S}(R_k)$ without computing whole spectrum.

Numerical Example 1

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

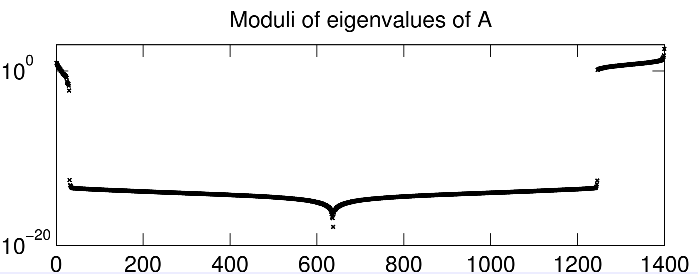
With tol $= 10^{-8}$, alg converges in 19 iterations to

$$X = \begin{bmatrix} 1.0000 & -0.8084 & 0.1916 & 0.1068 \\ -0.8084 & 1.0000 & -0.6562 & 0.1916 \\ 0.1916 & -0.6562 & 1.0000 & -0.8084 \\ 0.1068 & 0.1916 & -0.8084 & 1.000 \end{bmatrix}.$$

$\|A - X\|_F = 2.13$ and X has rank 3.

Numerical Example 2, from Finance

A: stock data, $n = 1399$. $a_{ii} \equiv 1$, $|a_{ij}| \leq 1$, but not psd.
A highly rank deficient with 1245 nonpositive ei'vals $\Rightarrow$ $\mathrm{rank}(X) \leq 154$.



tol $= 10^{-4}$, since data accurate to 2–3 sig figs only.
67 iterations, $\|A - X\|_F = 20.96$.
Athlon X2 4400 using NAG components: 8 minutes.

Newton Method

Qi & Sun (2006): quadratically convergent Newton method based on theory of **strongly semismooth matrix functions**.

- Applies Newton to **dual** of min $\|A - X\|$ problem.
- Dual problem is differentiable, but *not twice differentiable*.
- Cost per iteration:
  - One eigendecomposition.
  - Conjugate gradient method to solve one linear system.
- 10 iterations or less in tests.
- NAG implementation in progress.

Structured Correlation Problem 1

1-parameter correlation matrix

$$X(c) = \begin{bmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{bmatrix}.$$

For given A, nearest $X(c)$ in Frobenius norm given by

$$c = \frac{e^T A e - \mathrm{trace}(A)}{n^2 - n},$$

where $e = [1, 1, \ldots, 1]^T$.

Structured Correlation Problem 2

n-parameter correlation matrix:

$$A(x) = \mathrm{diag}(1 - x_i^2) + xx^T,$$

i.e., $a_{ij} = x_i x_j$, $i \neq j$.

Theorem (Higham & Raydan, 2006)

Let $x \in \mathbb{R}^n$ with $|x_i| \leq 1$ for all i. Then $\mathrm{rank}(A) = \min(p + 1, n)$, where p is the number of x_i for which $|x_i| < 1$.

Conclusions

- Feasible to compute **nearest** correlation matrix.
- Alternating projections
  - easy to implement,
  - guaranteed to find global minimum,
  - can exploit low rank solutions,
  - linearly convergent,
  - $O(n^3)$ flops per iteration and $O(n^2)$ storage.
- Newton method may be preferable.
- Algorithms for structured problems under development.
- NAG has relevant routines, with more imminent.

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