Functions of Matrices: From Cayley and Sylvester to MATLAB

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Term “matrix” coined in 1850 by James Joseph Sylvester, FRS (1814–1897).

Matrix algebra developed by Arthur Cayley, FRS (1821–1895).

Memoir on the Theory of Matrices (1858).
Cayley and Sylvester on Matrix Functions

- Cayley considered matrix square roots in his 1858 memoir.


- Sylvester gave first definition of $f(A)$ for a general $f$.

Matrices in Applied Mathematics

- Frazer, Duncan & Collar, Aerodynamics Division of NPL: aircraft flutter, matrix structural analysis.

- **Elementary Matrices & Some Applications to Dynamics and Differential Equations, 1938.** Emphasizes importance of $e^A$.

- Arthur Roderick Collar, FRS (1908–1986): "First book to treat matrices as a branch of applied mathematics".
Applications

- Monte-Carlo simulations in theoretical particle physics: $e^A$, $\log(A)$,
- Procrustes problems in factor and shape analysis: $(A^* A)^{1/2}$,
- Dynamical systems: $e^A$, $\log(A)$,
- Algebraic Riccati equations in control theory: $\text{sign}(A)$,

Markov models/finance

If $P$ transition matrix for 1 year, $P^{1/12} = e^{\frac{1}{12} \log P}$ is matrix for 1 month.
The most important matrix function, with many applications.

Vast literature.

Moler & Van Loan: Nineteen dubious ways to compute the exponential of a matrix, 1978.

“The availability of \texttt{expm}(A) in early versions of MATLAB quite possibly contributed to the system’s technical and commercial success.”

Matrix Exponential Formulae

\[
I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots
\]

\[
\lim_{s \to \infty} (I + A/s)^s
\]

\[
(e^{A/2s})^{2s}, \quad s \in \mathbb{Z}
\]

\[
\frac{1}{2\pi i} \int_{\Gamma} e^{z} (zI - A)^{-1} \, dz
\]

\[
\sum_{i=1}^{n} f[\lambda_1, \ldots, \lambda_i] (A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_{i-1})
\]

\[
Z \text{diag}(e^{J_k}) Z^{-1}
\]
Matrix Exponential Formulae

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\[ Z \text{diag}(e^{J_k}) Z^{-1} \]
Scaling and Squaring Method

“Least dubious of the original nineteen ways”.

- $B \leftarrow A/2^s$ so $\|B\|_\infty \approx 1$
- $r_m(B) = [m/m]$ Padé approximant to $e^B$
- $X = r_m(B)^{2^s} \approx e^A$

Two integer parameters: $s$ and $m$.

- Originates with Lawson (1967).
- H (2005) questioned existing choice of $s$ and $m$, as used in MATLAB’s `expm`.
- Using new analysis, with symbolic and high precision computation, able to derive optimal $s$ and $m$.
- New alg is faster: up to 32% speedup:

![Comparison of old and new algorithms](chart.jpg)

- **1000×1000 matrix**
  - `expm_old`: 3.67 secs
  - `expm_new`: 2.50 secs
New alg in MATLAB (\texttt{expm}) since Version 7.2 (R2006a), Mathematica (\texttt{MatrixExp}) since Version 5.1.

- New alg is also \textbf{more accurate}.
- Rare example of \textbf{faster implies more accurate} in numerical analysis.
- Reason: \textit{“fewer operations }\Rightarrow\textit{ smaller growth of errors”}—but latter implication is usually false!
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Growing body of work on matrix functions, stimulated by applications.

Of particular importance: $f(A)b$ or $u^*f(A)v$, for large, sparse $A$.


We’re still using notation/results of Cayley and Sylvester.
T. Crilly. 
*Arthur Cayley: Mathematician Laureate of the Victorian Age.* 

R. A. Frazer, W. J. Duncan, and A. R. Collar. 
*Elementary Matrices and Some Applications to Dynamics and Differential Equations.* 
J. Grabiner.

N. J. Higham.

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Book to appear.