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Functions of Matrices: From Cayley and Sylvester to MATLAB

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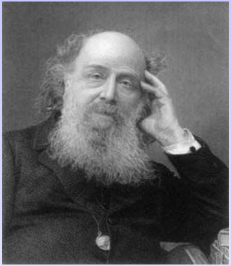
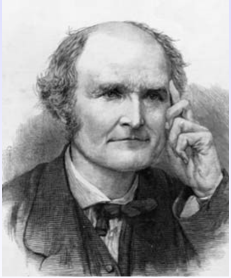
Cayley and Sylvester

- Term **"matrix"** coined in 1850 by James Joseph Sylvester, FRS (1814–1897).



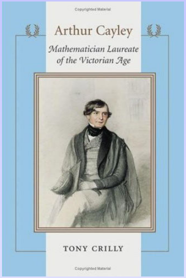
- **Matrix algebra** developed by Arthur Cayley, FRS (1821–1895).
  Memoir on the Theory of Matrices (1858).



Cayley and Sylvester on Matrix Functions

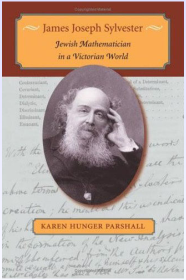
- Cayley considered matrix square roots in his 1858 memoir.

  Tony Crilly, *Arthur Cayley: Mathematician Laureate of the Victorian Age*, 2006.



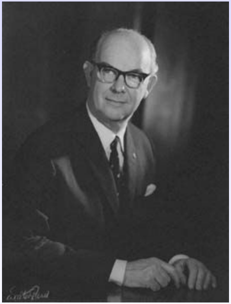
- Sylvester gave first definition of $f(A)$ for a general f.

  Karen Hunger Parshall, *James Joseph Sylvester. Jewish Mathematician in a Victorian World*, 2006.



Matrices in Applied Mathematics

- Frazer, Duncan & Collar, Aerodynamics Division of NPL: aircraft flutter, matrix structural analysis.
- **Elementary Matrices & Some Applications to Dynamics and Differential Equations, 1938**. Emphasizes importance of e^A.
- Arthur Roderick Collar, FRS (1908–1986): *"First book to treat matrices as a branch of applied mathematics".*



Applications

- Monte-Carlo simulations in theoretical particle physics: e^A, $\log(A)$,
- Procrustes problems in factor and shape analysis: $(A^*A)^{1/2}$,
- dynamical systems: e^A, $\log(A)$,
- algebraic Riccati equations in control theory: $\operatorname{sign}(A)$,

. . . .

Markov models/finance

If P transition matrix for 1 year, $P^{1/12} = e^{\frac{1}{12}\log P}$ is matrix for 1 month.

Matrix Exponential, e^A

- The most important matrix function, with many applications.
- Vast literature.
- Moler & Van Loan: **Nineteen dubious ways to compute the exponential of a matrix, 1978**.

"The availability of **`expm(A)`** *in early versions of MATLAB quite possibly contributed to the system's technical and commercial success."*

— **Cleve Moler (2003)**

Matrix Exponential Formulae

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$\lim_{s\to\infty} (I + A/s)^s$$

$$(e^{A/2^s})^{2^s}, \quad s \in \mathbb{Z}$$

$$\frac{1}{2\pi i} \int_\Gamma e^z (zI - A)^{-1} \, dz$$

$$\sum_{i=1}^{n} f[\lambda_1, \dots, \lambda_i](A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_{i-1})$$

$$Z \operatorname{diag}(e^{J_k}) Z^{-1}$$

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Scaling and Squaring Method

"Least dubious of the original nineteen ways".

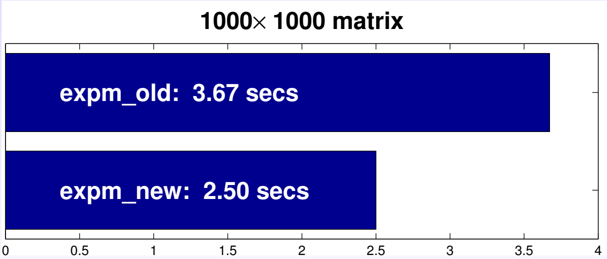
- $B \leftarrow A/2^s$ so $\|B\|_\infty \approx 1$
- $r_m(B) = [m/m]$ Padé approximant to e^B
- $X = r_m(B)^{2^s} \approx e^A$

Two integer parameters: s and m.

- Originates with **Lawson (1967)**.
- Analysis and alg by **Ward (1977)**, **Moler & Van Loan (1978)**.

Improved Scaling and Squaring Method (1)

- H (2005) questioned existing choice of *s* and *m*, as used in MATLAB's **`expm`**.
- Using new analysis, with symbolic and high precision computation, able to derive **optimal** *s* and *m*.
- New alg is faster: up to *32% speedup*:



Improved Scaling and Squaring Method (2)

New alg in MATLAB (`expm`) since Version 7.2 (R2006a), Mathematica (`MatrixExp`) since Version 5.1.

- New alg is also more accurate.
- Rare example of **faster implies more accurate** in numerical analysis.
- Reason: "*fewer operations* $\Rightarrow$ *smaller growth of errors*"—but latter implication is usually false!

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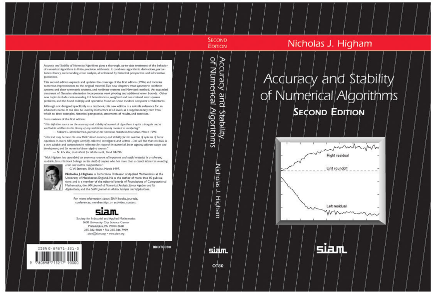

Summary

- Growing body of work on matrix functions, stimulated by applications.
- Of particular importance: $f(A)b$ or $u^*f(A)v$, for large, sparse A.
- My **Functions of Matrices: Theory and Computation (2008)** covers current state of the art.
- We're still using notation/results of Cayley and Sylvester.

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