J-Orthogonal Matrices: Properties and Generation

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J-Orthogonal Matrix

$Q \in \mathbb{R}^{n\times n}$ is J-*orthogonal* if

$$Q^T J Q = J,$$

where

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \qquad p + q = n.$$

Such matrices form a multiplicative group: the pseudo-orthogonal group.

Exchange Operator

$$y = \begin{array}{c} \\ p \\ q \end{array}\!\!\begin{array}{c} 1 \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{array} = \begin{array}{c} \\ p \\ q \end{array}\!\!\begin{array}{c} \quad p \qquad\quad q \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array}\!\!\begin{array}{c} \\ p \\ q \end{array} = Ax,$$

where A_{11} is nonsingular, can be rewritten

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \operatorname{exc}(A) \begin{bmatrix} y_1 \\ x_2 \end{bmatrix},$$

where the *exchange operator*,

$$\operatorname{exc}(A) = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.$$

Aka gyration operator, sweep operator, principal pivot transform. See survey by Tsatsomeros (2000).

Exchange Operator Properties

Theorem 1 *Let $A \in \mathbb{R}^{n\times n}$. If A is J-orthogonal then $\mathrm{exc}(A)$ is orthogonal. If A is orthogonal and A_{11} is nonsingular then $\mathrm{exc}(A)$ is J-orthogonal.*

Proof 1: with $z = \begin{bmatrix} y_1^T & x_2^T \end{bmatrix}^T$,

$$z^T J z = \|x_1\|^2 - \|y_2\|^2 = z^T \mathrm{exc}(A)^T J\, \mathrm{exc}(A) z.$$

Proof 2: use $\mathrm{exc}(JAJ) = J\mathrm{exc}(A)J = \mathrm{exc}(A^T)^T$, $\mathrm{exc}(A)^{-1} = \mathrm{exc}(A^{-1})$.

Theorem 2 *Let $A \in \mathbb{R}^{n\times n}$ with A_{11} nonsingular. Then $\mathrm{exc}(A) + \mathrm{exc}(A)^T$ is congruent to $A + A^T$.*

Hyperbolic CS Decomposition

Theorem 3 (Grimme, Sorensen & Van Dooren, 1996) *Let*

$$Q = \begin{matrix} & p & q \\ p & Q_{11} & Q_{12} \\ q & Q_{21} & Q_{22} \end{matrix}$$

be J*-orthogonal and assume that* $q \geq p$*. Then there are orthogonal* $U_1, V_1 \in \mathbb{R}^{p\times p}$ *and* $U_2, V_2 \in \mathbb{R}^{q\times q}$ *s.t.*

$$\begin{bmatrix} U_1^T & 0 \\ 0 & U_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{array}{c} \begin{matrix} p & p & q-p \end{matrix} \\ \left[\begin{array}{c|cc} C & -S & 0 \\ \hline -S & C & 0 \\ 0 & 0 & I_{q-p} \end{array}\right] \begin{matrix} p \\ p \\ q-p \end{matrix} \end{array},$$

where $C = \mathrm{diag}(c_i)$, $S = \mathrm{diag}(s_i)$ *and* $C^2 - S^2 = I$*.*

New proof: apply exchange operator to the standard CS decomposition!

Generating Random J-Orthogonal Matrices

To generate random $Q \in \mathbb{R}^{n\times n}$, assuming $q \ge p$ wlog:

1. Generate random orthogonal matrices $U_1, V_1 \in \mathbb{R}^{p\times p}$ and $U_2, V_2 \in \mathbb{R}^{q\times q}$ from Haar distribution.
2. Choose $C = \mathrm{diag}(c_i)$ and $S = \mathrm{diag}(s_i)$ according to constraints $C > S \ge 0$ and $C^2 - S^2 = I$.
3. Form $Q = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} C & -S & 0 \\ -S & C & 0 \\ 0 & 0 & I_{q-p} \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & V_2^T \end{bmatrix}$.

- Numerically stable construction.
- $\kappa_2(Q) = \|Q\|_2^2 = \max_{i=1:p}\left(c_i + \sqrt{c_i^2 - 1}\right)^2$.

J-Orthogonalization

Theorem 4 *If $A \in \mathbb{R}^{n\times n}$ and JA^TJA has no ei'vals on $\mathbb{R}^-$ then A has a unique indefinite polar decomposition $A = QS$, where Q is J-orthogonal and S is J-symmetric (SJ is symm) with ei'vals in open right half-plane.*

Lemma 1 *Let $A \in \mathbb{R}^{n\times n}$ have an indefinite polar decomposition $A = QS$. If $\|Q^{-1}(A-Q)\|_2 < 1$ then*

$$\frac{\|A^TJA - J\|_2}{\|A\|_2(\|A\|_2 + \|Q\|_2)} \le \frac{\|A-Q\|_2}{\|A\|_2} \le \frac{\|A^TJA - J\|_2}{\|A\|_2^2}\ \|A\|_2\|Q\|_2.$$

The lower bound always holds.

Iterations

Newton:

$$X_{k+1} = \tfrac{1}{2}(X_k + J X_k^{-T} J), \qquad X_0 = A.$$

$X_k \to Q$ quadratically.

Schulz:

$$X_{k+1} = \tfrac{1}{2} X_k (3I - J X_k^T J X_k), \qquad X_0 = A.$$

$X_k \to Q$ quadratically if $\|A^T J A - J\|_2 < 1$, since

$$R_{k+1} = \frac{3}{4} R_k^2 + \frac{1}{4} R_k^3, \qquad R_k = I - J X_k^T J X_k.$$

Conclusions

- ★ Exchange operator is valuable tool when dealing with J-orthogonal matrices.
- ★ Hyperbolic CSD is direct consequence of CSD!
- ★ Efficient alg for generating random J-orthogonal Q with specified $\kappa_2(Q)$.
- ★ Can J-orthogonalize using Newton or Schulz iterations.

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