Linear Algebra Meets Lie Algebra

The Kostant-Wallach Theory

Noam Shomron
Beresford Parlett

University of California, Berkeley

June, 2010

ILAS Meeting, Pisa, Italy

Linear Algebra Meets Lie Algebra

The Kostant-Wallach Theory

Noam Shomron
Beresford Parlett

University of California, Berkeley

June, 2010

ILAS Meeting, Pisa, Italy

Gelfand-Zeitlin theory from the perspective of classical mechanics, I and II
Studies in Lie Theory, 2006.

An Equivalence Relation on Square Matrices

$\mathcal{M}(n)$ = all $n \times n$ complex matrices with Poisson structure.
$B \in \mathcal{M}(n), B_j := B(1:j, 1:j)$

Definition (Ritz Values)

$\mathcal{R}(B) := (Eig(B_1), Eig(B_2), \ldots, Eig(B_n))$

$B \in \mathcal{M}(n), C \in \mathcal{M}(n)$,
$B \sim C \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(C)$.
? Gelfand Equivalence ?

An Equivalence Relation on Square Matrices

$\mathcal{M}(n)$ = all $n \times n$ complex matrices with Poisson structure.
$B \in \mathcal{M}(n), B_j := B(1:j, 1:j)$

Definition (Ritz Values)

$\mathcal{R}(B) := (Eig(B_1), Eig(B_2), \ldots, Eig(B_n))$

$B \in \mathcal{M}(n), C \in \mathcal{M}(n)$,
$B \sim C \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(C)$.
? Gelfand Equivalence ?

Given any generic multiset Λ of $\binom{n+1}{2}$ complex numbers then
$\mathcal{M}_\Lambda(n) := \{B \in \mathcal{M}(n) \quad | \quad \mathcal{R}(B) = \Lambda\}$,
a fibre of $\mathcal{M}(n)$ as a Lie Algebra.

Each $\mathcal{M}_\Lambda(n)$ is a symplectic leaf of $\mathcal{M}_\Omega(n)$.

Why study $\sim$?

Gil Strang

Kostant and Wallach found a "classical" analogue of the Gelfand - Kirilov theorem. They constructed a Lie Group which acts on $\mathcal{M}(n)$ and preserves Ritz values.

Lemma

Λ fixes the diagonal of each $B \in \mathcal{M}_\Lambda(n)$.

Proof. $B(j,j) = trace(B_j) - trace(B_{j-1})$.

Why study $\sim$?

Gil Strang

Kostant and Wallach found a "classical" analogue of the Gelfand - Kirilov theorem. They constructed a Lie Group which acts on $\mathcal{M}(n)$ and preserves Ritz values.

Lemma

Λ fixes the diagonal of each $B \in \mathcal{M}_\Lambda(n)$.

Proof. $B(j,j) = trace(B_j) - trace(B_{j-1})$.

Elementary Conjugations

(i) transposition: $B \to B^T$
(ii) diagonal similarity: $B \to DBD^{-1}$.

Lemma (K and W)

For any generic Λ,
$\mathcal{M}_\Lambda(n)$ contains exactly one unit upper **Hessenberg** *matrix.*

Generic Case

K and W found a “nice” set of coordinates to specify members of $\mathcal{M}_\Lambda(n)$ for generic Λ;
$s := (s^{(1)}, \ldots, s^{(n-1)}), s^{(j)} \in (\mathbb{C}^\times)^j$.

(Λ, s) are analogous to Darboux coordinates (q, p) in Hamilton-Jacobi theory of Mechanics.

Generic Case

K and W found a "nice" set of coordinates to specify members of $\mathcal{M}_\Lambda(n)$ for generic Λ;
$s := (s^{(1)}, \ldots, s^{(n-1)}), s^{(j)} \in (\mathbb{C}^x)^j$.

(Λ, s) are analogous to Darboux coordinates (q, p) in Hamilton-Jacobi theory of Mechanics.

Definition (Disjointness Conditions)

$(G1_j)$ elements of $Eig(B_j)$ are distinct

$(G2_j)$ $Eig(B_j) \cap Eig(B_{j-1}) = \emptyset$

Generic Case

K and W found a "nice" set of coordinates to specify members of $\mathcal{M}_\Lambda(n)$ for generic Λ;
$s := (s^{(1)}, \ldots, s^{(n-1)}), s^{(j)} \in (\mathbb{C}^\times)^j$.

(Λ, s) are analogous to Darboux coordinates (q, p) in Hamilton-Jacobi theory of Mechanics.

Definition (Disjointness Conditions)

($G1_j$) elements of $Eig(B_j)$ are distinct

($G2_j$) $Eig(B_j) \cap Eig(B_{j-1}) = \emptyset$

Definition

$\Lambda := (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$
$\Lambda_j :=$ either $j \times j$ invertible diagonal matrix or its diagonal entries in some **fixed order** .

$\mathcal{M}_\Omega(n) =$ the generic fibres in $\mathcal{M}(n)$.

The Dual Coordinates

$B \in \mathcal{M}_\Lambda(n) \subset \mathcal{M}_\Omega(n)$.
$(G1_m) \Rightarrow B_m = G_m \Lambda_m (G_m)^{-1}, \quad G_m \in GL(m)$.
G_m unique if last row is ones, by $(G2_m)$.

Definition

b_m and $c_m \in \mathbb{C}^m$, given by

$$\begin{aligned} B_{m+1} &= \begin{pmatrix} G_m & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \Lambda_m & c_m \\ {b_m}^T & \delta_{m+1} \end{pmatrix} \begin{pmatrix} {G_m}^{-1} & 0 \\ 0^T & 1 \end{pmatrix} \\ &= \begin{pmatrix} B_m & G_m c_m \\ {b_m}^T {G_m}^{-1} & \delta_{m+1} \end{pmatrix}. \end{aligned}$$

The Dual Coordinates

$B \in \mathcal{M}_\Lambda(n) \subset \mathcal{M}_\Omega(n)$.
$(G1_m) \Rightarrow B_m = G_m \Lambda_m (G_m)^{-1}, \quad G_m \in GL(m)$.
G_m unique if last row is ones, by $(G2_m)$.

Definition

b_m and $c_m \in \mathbb{C}^m$, given by

$$\begin{aligned} B_{m+1} &= \begin{pmatrix} G_m & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \Lambda_m & c_m \\ {b_m}^T & \delta_{m+1} \end{pmatrix} \begin{pmatrix} {G_m}^{-1} & 0 \\ 0^T & 1 \end{pmatrix} \\ &= \begin{pmatrix} B_m & G_m c_m \\ {b_m}^T {G_m}^{-1} & \delta_{m+1} \end{pmatrix}. \end{aligned}$$

Theorem (BNP)

$s = (1, {b_2}^T, {b_3}^T, \ldots, {b_{n-1}}^T)$ *determines* B.

What is G_m ?

The G recurrence

$Diag(k) :=$ all $k \times k$ invertible, diagonal, complex matrices.

Definition (Cauchy Matrix)

$D \in Diag(m), \quad E \in Diag(m+1),$
$Cauchy(D,E)_{ij} := (d_i - e_j)^{-1}.$

Eigenvectors of a (down) arrow matrix

$$\begin{pmatrix} \Lambda_m & c_m \\ {b_m}^T & \delta_{m+1} \end{pmatrix} = \begin{pmatrix} -diag(c_m)Cauchy(\Lambda_m, \Lambda_{m+1}) \\ \text{ones} \end{pmatrix} \cdot$$
$$\Lambda_{m+1} \cdot \Pi \begin{pmatrix} Cauchy(\Lambda_{m+1}, \Lambda_m)diag(b_m) & \text{ones} \end{pmatrix}$$

$\Pi \in Diag(m+1)$ and depends only on Λ.

$$B_{m+1} = G_{m+1}\Lambda_{m+1}{G_{m+1}}^{-1}.$$

G Recurrence cont.

So, recurrence is

$$G_1 = (1),$$
$$G_{m+1} = \begin{pmatrix} -G_m diag(c_m) Cauchy(\Lambda_m, \Lambda_{m+1}) \\ \text{ones} \end{pmatrix}, 1 \leq m < n.$$

Where is ${b_m}^T$?

$c = (1, {c_2}^T, \ldots, {c_{n-1}}^T)$ determine G_n and unique $B \in \mathcal{M}_\Lambda(n)$ via

$$B = G_n \Lambda_n {G_n}^{-1}.$$

Definition $\chi_m =$ char. poly. of B_m.

Lemma (BNP)

$diag(b_m) diag(c_m) = -\chi_{m+1}(\Lambda_m)({\chi_m}'(\Lambda_m))^{-1} =: \Sigma_m.$

Σ_m completely determined by Λ.

Blemish: Need $diag(b_m)$ invertible.

Definition

$b := 1 \oplus diag(b_2) \oplus \cdots \oplus diag(b_{n-1}) \in Diag(\binom{n}{2})$

The Group Action

How do we generate generic $\mathcal{M}_\Lambda$?
G_n, defined by the G recurrence, depends on (generic) Λ and b, so

Definition

$G^{(b)} := G_n \in GL(n)$

Each $B \in \mathcal{M}_\Lambda(n)$ is uniquely given by

$$B = G^{(b)}\Lambda_n(G^{(b)})^{-1}.$$

$Diag(\binom{n}{2})$ is a commutative group under matrix multiplication

and acts on $\mathcal{M}_\Lambda(n)$ via

$$b' \circ B = G^{(b'b)}\Lambda_n(G^{(b'b)})^{-1}.$$

- ones $\in Diag(\binom{n}{2})$ is the identity element.
- $G^{(\text{ones})}\Lambda_n(G^{(\text{ones})})^{-1}$ is the unique unit upper Hessenberg matrix in $\mathcal{M}_\Lambda(n)$.

Regularity

Theorem (K and W)

For generic B, $tril(B)$ and $triu(B)$ determine each other.

Since $B = LDU$ perhaps $tril(L)$ (or $triu(U)$) serve as dual coordinates?

$$B_{m+1} = \begin{pmatrix} B_m & v \\ u^T & \delta_{m+1} \end{pmatrix}$$

Need mild condition on B, u, v. Equivalent formulations follow.

- u^T is a cyclic vector for B_m
- $\begin{pmatrix} B_m \\ u^T \end{pmatrix}$ observable.
- $\begin{pmatrix} B_m & v \end{pmatrix}$ controllable.
- minimum polynomial of v for B_m has maximal degree.
- centralizer of $B_m = \{f(B_m) | f \in \mathbb{C}[\cdot]\}$.
- B_m **regular**. (does `not` mean invertible).

Hence $tril(B)$ `not` suitable as dual coordinate.

Blemishes

- The angle coordinates in b must not vanish.
- Need a **fixed**, but arbitrary, ordering for each Λ_k.

No smooth ordering for eigenvalues.

$$R(t) = \begin{pmatrix} 0 & \exp(2\pi i t) \\ 1 & 0 \end{pmatrix} = G(t)\Lambda(t)G(t)^{-1}.$$

$$\Lambda(t) = \begin{pmatrix} \exp(\pi i t) & 0 \\ 0 & -\exp(\pi i t) \end{pmatrix}.$$

BUT

$$\Lambda(0) \neq \Lambda(1),\; R(0) = R(1).$$

Classical Mechanics to Poisson Geometry

$$\mathbb{R}^{2n} \to (R) \qquad \mathcal{M}(n) \to \mathbb{C}$$

$$f = f(q, p) \qquad f = \text{ polynomial in matrix entries}$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \ \frac{dp}{dt} = -\frac{\partial H}{\partial q} \qquad [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{jk}$$

$$\alpha_{ij}(B) = b_{ij}$$

$$\{\alpha_{ij}, \alpha_{kl}\} = \delta_{jk}\alpha_{il} - \delta_{il}\alpha_{jk}$$

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \qquad \{f, g\} = \sum_{ij,kl}\{\alpha_{ij}, \alpha_{kl}\}\frac{\partial f}{\partial \alpha_{ij}}\frac{\partial g}{\partial \alpha_{kl}}$$

$$\xi_f \cdot g = \{f, g\} \qquad \xi_f \cdot g = \{f, g\}$$

Kostant-Wallach Theory

Inspired by Gelfand, K and W seek a maximal integrable system on $\mathcal{M}(n)$ as a Poisson manifold.
Notation. $\mathcal{P}(n) :=$ all polynomial functionals in entries of $n \times n$ matrix , e.g. trace, det.

$$\begin{aligned}\mathcal{P}(k)^{GL(k)} &= \{\text{all } f \in \mathcal{P}(k) \text{ invariant under similarity } \} \\ &= \{ \text{ all symmetric polynomials in the eigenvalues } \} \\ &= \{ \text{ all polynomials in } trace({B_k}^m), m \leq k\}\end{aligned}$$

Natural Embedding: $B_k \to B_k \oplus I_{n-k}$.

K-W theory (cont.)

Solution. Maximal subalgebra of $\mathcal{P}(n)$

$J(n) := \mathcal{P}(1)^{GL(1)}\mathcal{P}(2)^{GL(2)} \cdots \mathcal{P}(n)^{GL(n)} \subset \mathcal{P}(n)$.

Basis for $J(n)$. Typical B $\in \mathcal{M}_\Omega(n)$,

$f_1 = tr(B_1), f_2 = tr(B_2), f_3 = tr({B_2}^2), f_4 = tr(B_3), \cdots$.

Typical element of $J(n)$ is

$$\sum_{\mu_i \geq 0} c_\mu \, {f_1}^{\mu_1} \, {f_2}^{\mu_2} \, \cdots \, {f_N}^{\mu_N}, \; N = \binom{n}{2}, \; c_\mu \in \mathbb{C}.$$

Also need $f_{N+j} = tr(B^j), \; j = 1, \cdots, n$. Casimir functions!

K-W Theory (cont.)

Adjoint Orbit. $\mathcal{O}_B =$ similarity class of B.
$\mathcal{O}_B$ is **regular** if it has maximal dimension $n^2 - n$.
Each $\mathcal{O}_B$ is a symplectic leaf on $\mathcal{M}(n)$.

For $f \in \mathcal{P}(n), \xi_f \cdot g := \{f, g\}$, $g \in \mathcal{P}(n)$.

Theorem 1. For any $f \in J(n), \xi_f$ is globally integrable on $\mathcal{M}(n)$.

Theorem 2. If $\mathcal{O}_B$ is **strongly regular** then Hamiltonians $\{f_i\}$ form a completely integrable system on $\mathcal{O}_B$.

B strongly regular $\Leftrightarrow \{(\xi_{f_i})_B\}$ are linearly independent.

The Gelfand-Zeitlin Group Action

Integrate the vector fields ξ_{f_i} to obtain $\exp(q_i \xi_{f_i})$.

$$Ad(G)B = GBG^{-1}$$

$$\exp(q\,\xi_{tr(B_m)^k}) \cdot B = Ad[\exp(-q\,k(B_m)^{k-1}) \oplus \text{ones}]B$$

Here is the group A.

Theorem 3. $a = a(q) \in A$ is given by
$a = \exp(q_1 \xi_{f_1}) \exp(q_2 \xi_{f_2}) \cdots \exp(q_N \xi_{f_N})$.

Recall Ritz values $\mathcal{R}(B)$. $\mathcal{M}_{\mathcal{R}(B)}(n)$ is a fibre.

Theorem 4. For B generic $\mathcal{M}_{\mathcal{R}(B)}(n) = \{a \cdot B \mid a \in A\},$ a single orbit.

Theorem 5. Unit upper Hessenberg matrices are strongly regular.

Reconciliation

Theorem (Shomron)

$b = \exp(-q) = s$

BNP's angle coordinates b are identical to the dual coordinates s of K and W .

Systems of Polynomials

Orthogonal polynomials $\longleftrightarrow$ $3TR$ (1D).

CMV, Fiedler $\longleftrightarrow$ pentadiagonal

$$C = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -a_1 & -a_2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$F \sim C, \quad \mathcal{R}(F) = \mathcal{R}(C).$$

An amazing similarity transformation!

The magic similarity transformation

$$F = YCY^{-1}$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$