1. Show that the value of $f(A)$ given by the Jordan canonical form definition is independent of the particular Jordan canonical form that is used.

2. (a) Let $A \in \mathbb{C}^{n\times n}$ have an eigenvalue λ and corresponding eigenvector x. Show that $(f(\lambda), x)$ is a corresponding eigenpair for $f(A)$.

(b) Suppose A has constant row sums α, that is, $Ae = \alpha e$, where $e = [1, 1, \ldots, 1]^T$. Show that $f(A)$ has row sums $f(\alpha)$. Deduce the corresponding result for column sums.

3. The matrix

$$A = \begin{bmatrix} -2 & 2 & -2 & 4 \\ -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & -1 & 4 \end{bmatrix}$$

has minimal polynomial $\psi(t) = (t-1)^2(t-2)$. Find $\cos(\pi A)$.

4. Show using the polynomial interpolation and Cauchy integral formula definitions of $f(A)$ that $AB = BA$ implies $f(A)B = Bf(A)$.

5. (a) Prove the *World's Most Fundamental Matrix Equation*

$$(I + AB)A = A(I + BA)$$

Show that it is equivalent to $(AB)A = A(BA)$ and $(I + AB)^{-1} = I - A(I + BA)^{-1}B$

(b) Show that for any poly p,

$$p(AB)A = Ap(BA).$$

(c) Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$ and let $f(AB)$ and $f(BA)$ be defined. Show that

$$Af(BA) = f(AB)A.$$

6. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, with $m \geq n$, and assume that BA is nonsingular. Let f be defined on the spectrum of $\alpha I_m + AB$, and if $m = n$ let f be defined at α. Prove that

$$f(\alpha I_m + AB) = f(\alpha)I_m + A(BA)^{-1}(f(\alpha I_n + BA) - f(\alpha)I_n)B.$$

Can this formula be generalized to $f(D + AB)$ with $D \in \mathbb{C}^{m\times m}$ diagonal by "replacing αI by D"?

7. It is known that the tridiagonal Toeplitz matrix

$$T_n(c, d, e) = \begin{bmatrix} d & e & & \\ c & d & \ddots & \\ & \ddots & \ddots & e \\ & & c & d \end{bmatrix}.$$

has eigenvalues

$$d + 2(ce)^{1/2}\cos(k\pi/(n+1)), \qquad k = 1\colon n.$$

Consider

$$T_n = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}, \qquad \widetilde{T}_n = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}.$$

T_n is the second difference matrix and we know its eigenvalues. Obtain the eigenvalues of $\widetilde{T}_n$. Hint: use the previous question.

8. If A and B are $n \times n$ matrices does $ABAB = 0$ imply $BABA = 0$?

9. Let the power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ have radius of convergence r. Show that for $X, E \in \mathbb{C}^{n\times n}$ with $\|X\| < r$, the Fréchet derivative

$$L(X, E) = \sum_{i=1}^{\infty} a_i \sum_{j=1}^{i} X^{j-1} E X^{i-j}, \tag{1}$$

and hence that the Kronecker form $K(X)$ of the Fréchet derivative is given by

$$K(X) = \sum_{i=1}^{\infty} a_i \sum_{j=1}^{i} (X^T)^{i-j} \otimes X^{j-1}.$$

10. Show that $\operatorname{sign}(A) = A$ for any involutory matrix.

11. How are $\operatorname{sign}(A)$ and $\operatorname{sign}(A^{-1})$ related?

12. Can

$$A = \begin{bmatrix} -1 & 1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

be the sign of some matrix?

13. Computational. Write MATLAB code to implement this algorithm, and include options for each of the following scalings:

$$\text{determinantal scaling:} \quad \mu_k = |\det(X_k)|^{-1/n}, \tag{2}$$

$$\text{spectral scaling:} \quad \mu_k = \sqrt{\rho(X_k^{-1})/\rho(X_k)}, \tag{3}$$

$$\text{norm scaling:} \quad \mu_k = \sqrt{\|X_k^{-1}\|/\|X_k\|}. \tag{4}$$

Algorithm 1 (Newton algorithm for matrix sign function) *Given a nonsingular* $A \in \mathbb{C}^{n\times n}$ *with no pure imaginary eigenvalues this algorithm computes* $X = \operatorname{sign}(A)$ *using the scaled Newton iteration. Two tolerances are used: a tolerance* tol_cgce *for testing convergence and a tolerance* tol_scale *for deciding when to switch to the unscaled iteration.*

1 $X_0 = A$; scale = true
2 for $k = 1\colon\infty$
3     $Y_k = X_k^{-1}$
4   if scale
5   Set μ_k to one of the scale factors (2)–(4).
6   else
7         $\mu_k = 1$
8   end
9     $X_{k+1} = \frac{1}{2}(\mu_k X_k + \mu_k^{-1} Y_k)$
10     $\delta_{k+1} = \|X_{k+1} - X_k\|_F / \|X_{k+1}\|_F$
11   if scale = true and $\delta_{k+1} \le$ tol_scale, scale = false, end
12   if $\|X_{k+1} - X_k\|_F \le (\text{tol\_cgce}\|X_{k+1}\|/\|Y_k\|)^{1/2}$ or
       ($\delta_{k+1} > \delta_k/2$ and scale = false)
13   goto line 16
14   end
15 end
16 $X = X_{k+1}$

Carry out experiments to compare the different scale factors.

14. Open Problem. Obtain conditions that are necessary or sufficient—and ideally both—for $\kappa_{\exp}(A)$ to be large.

15. Open Problem. Prove the stability or otherwise of the scaling and squaring algorithm, by relating rounding errors in the squaring phase to the conditioning of the e^A problem.

16. Computational. An interesting application of the theory of matrix functions is to the Drazin inverse of $A \in \mathbb{C}^{n\times n}$, which can be defined as the unique matrix A^D satisfying $A^D A A^D = A^D$, $AA^D = A^D A$, $A^{k+1}A^D = A^k$, where k is the index of A (the index of a matrix is the dimension of the largest Jordan block where zero eigenvalue appears). If $A \in \mathbb{C}^{n\times n}$ has index k then it can be written

$$A = P \begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix} P^{-1},$$

where B is nonsingular and N is nilpotent of index k, and then

$$A^D = P \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Find the function f for which $A^D = f(A)$. Implement the Schur-Parlett algorithm to compute A^D.

17. Computational. Let $A \in \mathbb{C}^{m\times m}$, $U \in \mathbb{C}^{m\times k}$ and $V \in \mathbb{C}^{m\times k}$ where A is Hermitian and $m \gg k$. Find an efficient way to evaluate $\operatorname{trace}(U^T e^A V)$.

18. Computational. Compute $A^\alpha b$ by solving the initial value ODE

$$\frac{dy}{dt} = \alpha(A - I)[t(A-I) + I]^{-1} y, \quad y(0) = b.$$

Compare your results from different ODE solvers.

19. Computational. Download and install the Matrix Function Toolbox from `http://www.ma.man.ac.uk/~higham/mftoolbox`. Replace the function `funm_condest1` with the update function downloadable from `http://www.maths.manchester.ac.uk/~higham/misc/ggsss13.php`.

Design and carry out some experiments to compare the estimated condition number using (a) finite differences, (b) the complex step approximation, and (c) the exact Fréchet derivative computed using `expm_cond`, `logm_cond`, or your own code.

20. Computational. By experimenting with the MATLAB exponentiation operator A^t, $A \in \mathbb{C}^{n\times n}$, $t \in \mathbb{R}$, work out how matrix exponentiation is implemented.

When you observe an interesting property of numbers,
ask if perhaps you are not seeing, in the 1×1 *case,*
an interesting property of matrices.
— OLGA TAUSSKY, *How I became a torchbearer for matrix theory* (1988)

We share a philosophy about linear algebra:
we think basis-free,
we write basis-free,
but when the chips are down we close the office door and
compute with matrices like fury.
— IRVING KAPLANSKY, *Reminiscences [of Paul Halmos]* (1991)