On Sum-and-Distance Systems, Reversible Square Matrices and Divisor Functions

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(Joint with Sally L. Hill, Martin N. Huxley and Karl Michael Schmidt)

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Bohemian Matrices and Applications workshop
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Dame Kathleen Ollerenshaw (1912-2014)

Dame Kathleen Ollerenshaw, was a noted mathematician and educationist. She also served as lord mayor of Manchester and was a local councillor in her native city for more than 25 years.

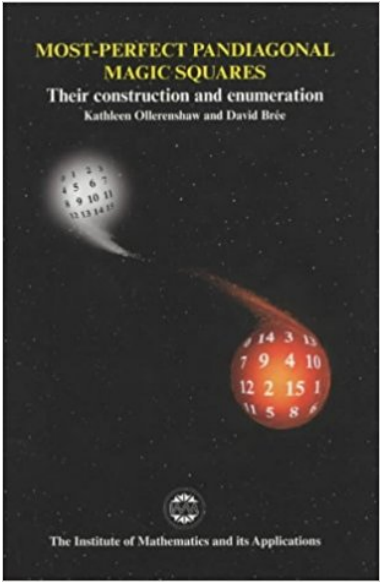
Today the University of Manchester offers their flagship *Dame Kathleen Ollerenshaw Research Fellowships* for outstanding early career research scientists and engineers.

Most-Perfect Pandiagonal Magic Squares

In mathematics, one of Dame Kathleen's particular interests was magic squares.

Kathleen's best known mathematical work, Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration (1998), co-authored with David Brée, was the result of her investigations.

They enumerated all (doubly-even) $n \times n$ most-perfect square matrices via a bijection between the $n \times n$ most-perfect squares and the $n \times n$ reversible squares which we now define.



Reversible Square Matrices

Definition

A *reversible square matrix*, $R = (r_{i\,j})_{n\times n}$ satisfies the three symmetry conditions:

- *reverse row similarity:* $r_{i\,j} + r_{i\;n+1-j} = r_{i\,j'} + r_{i\;n+1-j'}$,
- *reverse column similarity:* $r_{i\,j} + r_{n+1-i\;j} = r_{i'\,j} + r_{n+1-i'\;j}$,
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We call a square matrix a *principal reversible square* if all column (up-down) and row (left-right) entries are in ascending order and the entries $r_{1\,1}$ and $r_{1\,2}$ are 0 and 1.

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We call a square matrix a *principal reversible square* if all column (up-down) and row (left-right) entries are in ascending order and the entries $r_{1\,1}$ and $r_{1\,2}$ are 0 and 1. There are three 4×4 principal reversible squares

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 8 & 9 \\ 2 & 3 & 10 & 11 \\ 4 & 5 & 12 & 13 \\ 6 & 7 & 14 & 15 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 8 & 9 & 12 & 13 \\ 10 & 11 & 14 & 15 \end{pmatrix}$$

The $\mathcal{X}$-Factor

Let

$$\mathcal{X}_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \qquad \mathcal{X}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathcal{X}_{2n+1} = \begin{pmatrix} \frac{\mathcal{I}_n}{\sqrt{2}} & 0_n & \frac{\mathcal{J}_n}{\sqrt{2}} \\ 0_n^T & 1 & 0_n^T \\ \frac{\mathcal{J}_n}{\sqrt{2}} & 0_n & -\frac{\mathcal{I}_n}{\sqrt{2}} \end{pmatrix} \in \mathbb{R}^{(2n+1)\times(2n+1)}.$$

(The matrix $\mathcal{X}_{2n+1}$ turns into $\mathcal{X}_{2n}$ when its central row and column are deleted.)

Clearly $\mathcal{X}_m^2 = \mathcal{I}_m$ and $\mathcal{X}_m^T = \mathcal{X}_m$, so $\mathcal{X}_m$ is an orthogonal symmetric involution.

Conjugation of a reversible square with the matrix $\mathcal{X}_m$ gives rise to a block representation.

Reversible Square Block Representations

The block representations $\mathcal{X}_4 R \mathcal{X}_4$ for the three 4×4 principal reversible squares are given by

$$\begin{pmatrix} 15 & 15 & -1 & -3 \\ 15 & 15 & -1 & -3 \\ -4 & -4 & 0 & 0 \\ -12 & -12 & 0 & 0 \end{pmatrix} \begin{pmatrix} 15 & 15 & -7 & -9 \\ 15 & 15 & -7 & -9 \\ -2 & -2 & 0 & 0 \\ -6 & -6 & 0 & 0 \end{pmatrix} \begin{pmatrix} 15 & 15 & -3 & -5 \\ 15 & 15 & -3 & -5 \\ -6 & -6 & 0 & 0 \\ -10 & -10 & 0 & 0 \end{pmatrix}$$

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The entries in the blocks correspond to the three $2+2$ set pairs which we will shortly see correspond to the three non-inclusive sum-and-distance systems

$$\{\{1,3\},\{4,12\}\}, \quad \{\{7,9\},\{2,6\}\}, \quad \{\{3,5\},\{6,10\}\}$$

Reversible Square Block Representation Theorem

Block Represenation Theorem

Let $M \in \mathbb{R}^{n\times n}$. Then M is an $n \times n$ reversible square matrix if and only if there are vectors $a, b \in \mathbb{R}^{\nu}$ such that

$$M = \mathcal{X}_n \begin{pmatrix} 2w\mathcal{E}_\nu & 1_\nu a^T \\ b1_\nu^T & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n$$

if $n = 2\nu$ is even,

$$M = \mathcal{X}_n \begin{pmatrix} 2w\mathcal{E}_\nu & \sqrt{2}w\, 1_\nu & 2\, 1_\nu a^T \\ \sqrt{2}w\, 1_\nu^T & w & \sqrt{2}a^T \\ 2\, b1_\nu^T & \sqrt{2}b & \mathcal{O}_\nu \end{pmatrix} \mathcal{X}_n$$

if $n = 2\nu + 1$ is odd.

On $m + m$ Sum-and-Distance Systems

A sum-and-distance system expresses the first n integers of an arithmetic progression as the elements of a sum-and-distance set; that is, one comprised of all the sums and distances between the elements of two disjoint sets A and B.

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Definition

Let $m \in \mathbb{N}$ and consider positive integers $a_j, b_j \in \mathbb{N}$ ($j \in \{1, \ldots, m\}$) such that

$$a_1 < a_2 < \cdots < a_m, \qquad b_1 < b_2 < \cdots < b_m.$$

We call $\{\{a_j : j \in \{1, \ldots, m\}\}, \{b_j : j \in \{1, \ldots, m\}\}\}$
a) an $m + m$ (non-inclusive) sum-and-distance system if

$$\{a_j + b_k, |a_j - b_k| : j, k \in \{1, \ldots, m\}\} = \{1, 3, 5, \ldots, 4m^2 - 1\};$$

b) an $m + m$ inclusive sum-and-distance system if

$$\{a_j, b_k, a_j + b_k, |a_j - b_k| : j, k \in \{1, \ldots, m\}\} = \{1, 2, 3, \ldots, 2m(m + 1)\}.$$

The $3+3$ (non-inclusive) Sum-and-Distance Systems

Example

For $m = 3$ there are the seven $3+3$ (non-inclusive) sum-and-distance systems

$$\{\{1,3,5\},\{6,18,30\}\},\quad \{\{1,7,9\},\{2,22,26\}\},\quad \{\{1,11,13\},\{14,18,22\}\},$$
$$\{\{1,23,25\},\{2,6,10\}\},\quad \{\{3,9,15\},\{16,18,20\}\},\quad \{\{3,21,27\},\{4,6,8\}\},$$
$$\{\{7,9,11\},\{12,18,24\}\},$$

but just the one inclusive $3+3$ sum-and-distance system $\{\{1,2,3\},\{7,14,21\}\}$.

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but just the one inclusive $3+3$ sum-and-distance system $\{\{1,2,3\},\{7,14,21\}\}$.

If one considers the sums of the squares of the elements of the $3+3$ sum-and-distance systems, then an invariant property becomes apparent such that

$$1^2 + 3^2 + 5^2 + 6^2 + 18^2 + 30^2 = \ldots = 7^2 + 9^2 + 11^2 + 12^2 + 18^2 + 24^2 = 1295.$$

The Invariance of the Sum of Squares

Theorem 1

Let $m \in \mathbb{N}$ and $\{\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_m\}\}$ a (non-inclusive or inclusive) sum-and-distance system. Then

$$\sum_{j=1}^{m}(a_j^2 + b_j^2) = \begin{cases} \frac{1}{3!}(2m)((2m)^4 - 1) & \text{in the non-inclusive case,} \\ \frac{1}{4!}(2m+1)((2m+1)^4 - 1) & \text{in the inclusive case.} \end{cases}$$

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Proof.

In the non-inclusive case we use the formula

$$\sum_{j=1}^{n}(2j-1)^2 = \frac{n(4n^2-1)}{3} \qquad (n \in \mathbb{N})$$

to find

$$2m\sum_{j=1}^{m}(a_j^2+b_j^2) = \sum_{j=1}^{m}\sum_{k=1}^{m}((a_j+b_k)^2+(a_j-b_k)^2) = \sum_{j=1}^{2m^2}(2j-1)^2 = \frac{1}{6}\,4m^2(16m^4-1).$$

□

Bijections and Enumerations

Theorem 2

Let $m \in \mathbb{N}$. Then there is a bijection between the $m + m$ non-inclusive sum-and-distance systems and the $2m \times 2m$ principal reversible squares, and there is a bijection between the $m + m$ inclusive sum-and-distance systems and the $(2m + 1) \times (2m + 1)$ principal reversible squares.

Bijections and Enumerations

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The enumeration of all principal reversible square matrices of side length $n = 2^r p^s$, was determined by Ollerenshaw and Brée in 1998. The case for any side length n is given below.

Theorem 3

Let $n = \prod_{k=1}^{t} p_k^{a_k}$. Then the number of different $n \times n$ principal reversible squares N_n, and so sum-and-distance systems of the corresponding type, is given by

$$N_n = \sum_{j=1}^{\Omega(n)} \sum_{l=1}^{j} \sum_{m=0}^{j} (-1)^{l+m} \binom{j}{l} \binom{j}{m} \prod_{k=1}^{t} \binom{a_k + l - 1}{l - 1} \binom{a_k + m}{m},$$

where $\Omega(n) = a_1 + a_2 + \ldots + a_t$, counts the number of prime factors of n including repeats.

The j-th Divisor Function $d_j(n)$

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The jth divisor function $d_j(n)$, which counts the ordered factorisations of a positive integer n into j positive integer factors, is a very well-known arithmetic function.

For example

$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4 = 4 \times 3 = 6 \times 2 = 12 \times 1,$$

so that $d_2(12) = 6$.

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The divisor function lies at the heart of a number of open number theoretical problems, e.g. the *additive divisor problem* of finding the asymptotic of

$$\sum_{n \leq x} d_j(n)\, d_j(n+h) \tag{1}$$

for large x, which is notoriously difficult if $j \geq 3$.

Properties of $d_j(n)$

The j-th divisor is an arithemtic multiplicative function that has many properties of interest.

Lemma 1

- *Let $j, n \in \mathbb{N}$, $j \geq 2$. Then the jth divisor function $d_j(n)$ satisfies the sum-over-divisors recurrence relation*

$$d_j(n) = \sum_{m|n} d_{j-1}(m).$$

- *Let $p_1, \ldots, p_t$ be distinct primes, $t \in \mathbb{N}$. Then, for any $j \in \mathbb{N}$.*

$$d_j(p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}) = \prod_{k=1}^{t} \binom{a_k + j - 1}{a_k} \quad (a_1, \ldots, a_t \in \mathbb{N}_0).$$

- *Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Then for $j \in \mathbb{N}$, the divisor function d_j has the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s} = \zeta(s)^j.$$

For example $\sum_{n=1}^{\infty} \frac{d_j(n)}{n^6} = \zeta(6)^j = \left(\frac{\pi^6}{945}\right)^j$

The j-th Non-Trivial Divisor Function $c_j(n)$

We consider the rather less well-studied jth *non-trivial divisor function* $c_j(n)$, which counts the ordered proper factorisations of a positive integer n into j factors, each of which is greater than or equal to 2.

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While $d_j(n)$, for given n, is obviously monotone increasing in j, since factors of 1 can be freely introduced, $c_j(n)$ will shrink back to 0 as j gets too large, and indeed $c_j(n) = 0$ if $n < 2^j$.

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For example

$$12 = 2 \times 6 = 3 \times 4 = 4 \times 3 = 6 \times 2,$$

and

$$12 = 2 \times 2 \times 3 = 2 \times 3 \times 2 = 3 \times 2 \times 2,$$

so that $c_2(12) = 4$, $c_3(12) = 3$ but $c_4(12) = 0$.

Properties of the j-th Non-Trivial Divisor Function $c_j(n)$

The arithmetic function c_j satisfies a sum-over-divisors recurrence with respect to j analogous to, but subtly different from, that given for d_j in Lemma 1.

Lemma 2

- *Let $j, n \in \mathbb{N}$, $j \geq 2$. Then the jth non-trivial divisor function satisfies the sum-over-divisors recurrence relation*

$$c_j(n) = \sum_{\substack{m|n \\ m<n}} c_{j-1}(m) = \sum_{\substack{m|n \\ m\notin\{1,n\}}} c_{j-1}(m).$$

- *For $j \in \mathbb{N}$, the non-trivial divisor function c_j has the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{c_j(n)}{n^s} = (\zeta(s) - 1)^j \, .$$

We remark that, unlike $d_j(n)$, $c_j(n)$ is *not* a multiplicative arithmetic function. For example, $(2, 5) = 1$, and yet $c_2(10) = 2 \neq 0 \times 0 = c_2(2)c_2(5)$.

The Associated Divisor Function $c_j^{(r)}(n)$

Additionally we define the *associated divisor function* $c_j^{(r)}(n)$, for $r \in \mathbb{N}_0$, by

$$c_j^{(0)}(n) = c_j(n), \qquad c_j^{(r)}(n) = \sum_{m|n} c_j^{(r-1)}(m) \quad (n, r \in \mathbb{N}).$$

This definition is motivated by the sum-over divisors recurrence for $d_j(n)$

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Lemma 3

For $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$, the Dirichlet series of the associated divisor function $c_j^{(r)}$ is given by

$$\sum_{n=1}^{\infty} \frac{c_j^{(r)}(n)}{n^s} = \zeta(s)^r (\zeta(s) - 1)^j.$$

We also have a binomial form for the value of $c_j^{(r)}(n)$ at prime powers; this is somewhat analogous to that given in Lemma 1, but note that the present function is not multiplicative.

Lemma 4

Let $j, a \in \mathbb{N}$, $r \in \mathbb{N}_0$, and p a prime. Then

$$c_j^{(r)}(p^a) = \binom{a + r - 1}{j + r - 1}.$$

Divisor Function Expressions and N_n

Theorem 4

Let $m \in \mathbb{N}$. Then there are

$$N_{2m} = \sum_{j=1}^{\Omega(2m)} c_j(2m)\left(c_j(2m) + c_{j+1}(2m)\right) = \sum_{j=1}^{\Omega(2m)} c_j^{(0)}(2m)\, c_j^{(1)}(2m)$$

different $m+m$ non-inclusive sum-and-distance systems, and

$$N_{2m+1} = \sum_{j=1}^{\Omega(2m+1)} c_j(2m+1)\left(c_j(2m+1)+c_{j+1}(2m+1)\right) = \sum_{j=1}^{\Omega(2m+1)} c_j^{(0)}(2m+1)\, c_j^{(1)}(2m+1)$$

different $m+m$ inclusive sum-and-distance systems.

Divisor Function Expressions and N_n

Lemma 5

Let $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Then the associated divisor function $c_j^{(r)}(n)$ can be expressed in terms of the divisor function $d_j(n)$ as follows,

$$c_j^{(r)}(n) = \sum_{i=0}^{r} \binom{r}{i} c_{j+i}(n) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+r}(n) \qquad (n \in \mathbb{N}).$$

Remark

Combining Lemma 5 with Theorem 4 we have that the count N_n can be expressed in terms of the (multiplicative) divisor functions,

$$N_n = \sum_{j=1}^{\Omega(n)} \sum_{l=1}^{j} \sum_{m=0}^{j} (-1)^{l+m} \binom{j}{l} \binom{j}{m} d_l(n) d_{m+1}(n).$$

We note that the terms of the above sum bear some similarity to the sum underlying the additive divisor problem.

Prime Numbers and N_n

Corollary 1

Let $n \in \mathbb{N}$. Then $N_n = 1$ if and only if n is prime.

Proof.

If n is prime, then $c_1(n) = 1$ and $c_j(n) = 0$ for all $j \geq 2$, and it follows that $N_n = c_1(n)\,(c_1(n) + c_2(n)) = 1$. Conversely, suppose $n \geq 2$ is an integer such that $N_n = 1$. By Theorem 4,

$$N_n = \sum_{j=1}^{\Omega(n)} c_j(n)^2 + \sum_{j=1}^{\Omega(n)} c_j(n)c_{j+1}(n).$$

As all terms of these sums are non-negative and $c_1(n) = 1$, the total can equal 1 only if $c_j(n) = 0$ ($j \geq 2$), which implies that n is prime. □

Corollary 2

Let $n = p^a$ with $a \in \mathbb{N}$ and prime number p. Then

$$N_n = \binom{2a-1}{a}.$$

(The result follows by combinatorial identity (3.20) of [1].)

Ratios of Divisor Functions

Theorem 5

Let $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$, and suppose $n \in \mathbb{N}$ has prime factorisation $n = p_1^{a_1} \cdots p_t^{a_t}$. Then

$$\frac{c_j^{(r)}(n)}{d_{j+r}(n)} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\binom{j+r-1}{i}^t}{\prod_{k=1}^t \binom{a_k+j+r-1}{i}} = {}_{t+1}F_t(\{1-j-r\}_{i=1}^t, -j; \{1-a_i-j-r\}_{i=1}^t; 1). \quad (2)$$

Also, for $r \geq 1$

$$\frac{c_j^{(r)}(n)}{d_r(n)} = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \frac{\prod_{k=1}^t \binom{a_k+i+r-1}{i}}{\binom{i+r-1}{i}^t} = (-1)^j \; {}_{t+1}F_t(\{a_k+r\}_{k=1}^t, -j; \{r\}_{k=1}^t; 1). \quad (3)$$

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Thank you for listening!

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