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Exploiting Tropical Algebra in Numerical Linear Algebra

Françoise Tisseur
School of Mathematics
The University of Manchester

Joint work with

J. Hook (Bath), J. Pestana (Strathclyde),
M. Van Barel (Leuven),
L. Grammont, J. Hogg, V. Noferini, J. Scott, M. Sharify.

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What Is Tropical Algebra?

By "**tropical**" we refer to a semiring in which the addition operation is **min** or **max**.

In this talk, we consider the **max-plus semiring** $\mathbb{R}_{\max} = (\overline{\mathbb{R}}, \oplus, \otimes)$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$,

$$a \oplus b = \max(a, b), \quad a \otimes b = a + b, \quad \forall a, b \in \overline{\mathbb{R}},$$

and additive and multiplicative identities $-\infty$ and 0:

$$a \oplus -\infty = a, \quad a \otimes 0 = a.$$

(James Hook used the min-plus semiring in his talk.)

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Tropical algebra is the tropical analogue of linear algebra, working with matrices with entries in $\overline{\mathbb{R}}$. If $A, B \in \overline{\mathbb{R}}^{n \times n}$,

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \qquad (A \otimes B)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}.$$

Valuation

A **valuation** is a map from a field $\mathbb{F} \to \overline{\mathbb{R}}$ (provide a measure of of the size or multiplicity of elements of $\mathbb{F}$).

- $x \in \mathbb{C} \mapsto \mathcal{V}_c(x) = \log|x| \in \mathbb{R}_{\max} \quad (\log 0 = -\infty)$.
- For $A \in \mathbb{C}^{n\times n}$, $\mathcal{V}_c(A) = (\log|a_{ij}|)$.
- For $x, y \in \mathbb{C}$,

$$\mathcal{V}_c(xy) = \mathcal{V}_c(x) + \mathcal{V}_c(y),$$

  and when $|x| \gg |y|$ or $|x| \ll |y|$ then

$$\mathcal{V}_c(x+y) \approx \max\{\mathcal{V}_c(x), \mathcal{V}_c(y)\}.$$

How Can Tropical Algebra Help NLA?

- Some NLA problems are easier to study/solve when expressed in the tropical algebra setting, e.g.,
  - characterization of all **Hungarian scalings** of a matrix A.
- Tropical analogues of NLA problems offer order of magnitude approximation, e.g., for
  - entries in the **LU factors** or Cholesky factors of a matrix A,
  - roots of scalar polynomials and eigenvalues of matrices/**matrix polynomials**.

These approximations are often cheap to compute and useful for the design of preprocessing steps.

Hungarian Scaling

A two-sided diagonal scaling of $A \in \mathbb{C}^{n \times n}$ applied along with permutation P to $Ax = b$:

$$H = PD_1AD_2, \quad Hy = PD_1b, \quad x = D_2y,$$

with D_1, D_2 diagonal and $H = (h_{ij})$ s.t. $|h_{ij}| \leq 1$, $|h_{ii}| = 1$.

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- Improves stability of LU fact. with no pivoting [Olschowka & Neumaier'96], [Hogg & Scott'14]
- Effective preprocessing step for preconditioned iterative methods. [Benzi, Haws & Tůma'00]
- Code MC64 in HSL. [Duff & Koster'01]
  Worst case complexity: $O(n\tau + n^2\log(n))$, $\tau = nnz(A)$.
  In practice, complexity is $O(\tau)$.

Hungarian Pairs

Max-plus permanent of $\mathcal{A} \in \overline{\mathbb{R}}^{n \times n}$:

$$\operatorname{perm}(\mathcal{A}) = \bigoplus_{\pi \in \Pi_n} \bigotimes_{j=1}^{n} a_{\pi(j)j} = \max_{\pi \in \Pi_n} \sum_{j=1}^{n} a_{\pi(j)j}, \qquad (1)$$

Π_n: set of permutations of $\{1, \ldots, n\}$. Permutation π attaining max in (1) is an **optimal assignment**.

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Express (1) as a **linear programming problem** (LPP)

$$\mathrm{perm}(\mathcal{A}) = \max\Big\{\sum_{i,j=1}^{n} a_{ij}d_{ij} : d_{ij} > 0, \sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} d_{ji} = 1 \ \forall i\Big\}$$

with **dual problem**

$$\mathrm{perm}(\mathcal{A}) = \min\Big\{\sum_{i=1}^{n} u_i + v_i : u, v \in \mathbb{R}^n, a_{ij} - u_i - v_j \le 0\Big\}. \qquad (2)$$

A **Hungarian pair** is a solution (u, v) to (2).

Set of All Hungarian Pairs

Let π and (u, v) be an opt assignment and a Hungarian pair of $\mathcal{A} = \mathcal{V}_c(A)$ with $A \in \mathbb{C}^{n\times n}$. Then

$$H = P_\pi \mathrm{diag}_0\big(\exp(-u)\big) A \mathrm{diag}_0\big(\exp(-v)\big) \in \mathbb{C}^{n\times n}$$

is a **Hungarian scaling** of A.

Theorem (Hook, Pestana, T., Hogg'17)

*The **set of all Hungarian pairs of** $\mathcal{A}$ is given by*

$$\mathrm{Hung}(\mathcal{A}) = \big\{(u + s_{\pi^{-1}}, v - s) : s \in \mathrm{col}(\mathcal{H}^\star) \cap \mathbb{R}^n\big\},$$

where $\mathcal{H} = \mathcal{P}_\pi \otimes \mathrm{diag}_\infty(-u) \otimes \mathcal{A} \otimes \mathrm{diag}_\infty(-v)$ *and* $(s_{\pi^{-1}})_i = s_{\pi^{-1}(i)}$.

Here $\mathcal{H}^\star = \mathcal{I} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots \oplus \mathcal{H}^{n-1}$ is the **Kleene star** and $\mathrm{col}(\mathcal{H}) := \{\mathcal{H} \otimes x : x \in \overline{\mathbb{R}}^n\}$ is the column space of $\mathcal{H}$.

Summary/Comments

We have

- used max-plus algebra to characterize set of all Hungarian scalings for a given $A \in \mathbb{C}^{n \times n}$,
- shown that **max-balancing** a Hungarian scaled matrix yields the “**most diagonally dominant**" Hungarian scaled matrix with respect to some ordering.
- For max-balanced Hungarian scaled matrices, numerical experiments show
  - reduced need for row interchanges in GEPP,
  - improved stability of LU with no pivoting,
  - improved convergence rate of iterative methods.

(See Hook, Pestana, Tisseur, Hogg, MIMS Eprint 2015.36)

Objectives

Given $A \in \mathbb{C}^{n\times n}$, sparse with nonzero entries that vary widely in magnitude,

- approximate efficiently the order of magnitude of the entries in the LU factors of A,
- use large entries to define pattern matrix for an ILU preconditioner.

Objectives

Given $A \in \mathbb{C}^{n \times n}$, sparse with nonzero entries that vary widely in magnitude,

- approximate efficiently the order of magnitude of the entries in the LU factors of A,
- use large entries to define pattern matrix for an ILU preconditioner.

Use **max-plus algebraic techniques**: transform A into a max-plus matrix using valuation $\mathcal{V}_c : \mathbb{C} \mapsto \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$,

$$\mathcal{V}_c(x) = \log |x|, \quad (\log 0 = -\infty).$$

Basis for Approximation

- Entries in L and U can be expressed explicitly in terms of determinants of submatrices of $A \in \mathbb{C}^{n\times n}$, e.g.,

$$l_{ik} = \det\big(A([1:k-1, i], 1:k)\big) / \det\big(A(1:k, 1:k)\big), \; i \geq k,$$

- when A has large variation in the size of its entries,

$$\boxed{\mathcal{V}_c\big(\det(A)\big) \approx \operatorname{perm}\big(\mathcal{V}_c(A)\big)}, \qquad \textbf{(Heuristic 1)}$$

$\mathcal{V}_c : \mathbb{C} \mapsto \overline{\mathbb{R}}$, $\mathcal{V}_c(x) = \log|x|$, $(\log 0 = -\infty)$,
perm is the max-plus permanent, i.e., for $\mathcal{A} \in \overline{\mathbb{R}}^{n\times n}$,

$$\operatorname{perm}(\mathcal{A}) = \max_{\pi\in\Pi(n)} \sum_{i=1}^{n} a_{i,\pi(i)}.$$

Example

Let $\mathcal{V}_c(x) = \log_{10}|x|$ and consider

$$A = \begin{bmatrix} 10 & 0 & 1000 \\ 1 & 10 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \det(A) = -900.$$

Then

$$\mathcal{A} = \mathcal{V}_c(A) = \begin{bmatrix} 1 & -\infty & 3 \\ 0 & 1 & -\infty \\ -\infty & 0 & 0 \end{bmatrix},$$

$$\text{perm}(\mathcal{A}) = \max\{1+1+0, 3+0+0\} = 3,$$

which provides an approximation of $\log_{10}|\det(A)| \approx 2.95$.

Max-Plus LU Factors of $\mathcal{A} \in \overline{\mathbb{R}}^{n \times n}$

Let $\mathcal{L} = (\ell_{ij})$ and $\mathcal{U} = (u_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ be such that
$\ell_{ik} = u_{kj} = -\infty$ if $i, j < k$, and for $i, j \geq k$,

$$\ell_{ik} = perm\big(\mathcal{A}([1:k-1, i], 1:k)\big) - \text{perm}\big(\mathcal{A}(1:k, 1:k)\big),$$

$$u_{kj} = \text{perm}\big(\mathcal{A}(1:k, [1:k-1, j])\big) - \text{perm}\big(\mathcal{A}(1:k-1, 1:k-1)\big).$$

As a consequence of Heuristic 1 we have

Heuristic 2: If $\mathcal{A} = \mathcal{V}(A) \in \overline{\mathbb{R}}^{n \times n}$ has max-plus LU factors $\mathcal{L}, \mathcal{U} \in \overline{\mathbb{R}}^{n \times n}$ then $A \in \mathbb{C}^{n \times n}$ has LU fact $A = LU$ with

$$\mathcal{V}_c(L) \approx \mathcal{L}, \quad \mathcal{V}_c(U) \approx \mathcal{U}.$$

Example (Cont.)

The matrix $A = \begin{bmatrix} 10 & 0 & 1000 \\ 1 & 10 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has LU fact

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 1 & 0 \\ 0 & 0.1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 & 1000 \\ 0 & 10 & -100 \\ 0 & 0 & 11 \end{bmatrix}$$

and

$$\mathcal{A} = \mathcal{V}_c(A) = \begin{bmatrix} 1 & -\infty & 3 \\ 0 & 1 & -\infty \\ -\infty & 0 & 0 \end{bmatrix}$$

has max-plus LU factors

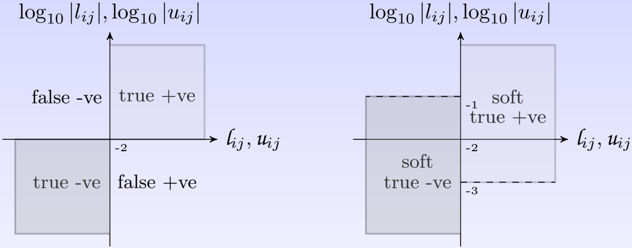
$$\mathcal{L} = \begin{bmatrix} 0 & -\infty & -\infty \\ -1 & 0 & -\infty \\ -\infty & -1 & 0 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} 1 & -\infty & 3 \\ -\infty & 1 & 2 \\ -\infty & -\infty & 1 \end{bmatrix}.$$

Note that $\mathcal{V}_c(L) = \mathcal{L}$ and $\mathcal{V}_c(U) \approx \mathcal{U}$.

Quality of Max-plus LU Approximation

233 matrices from U. Florida sparse matrix collection.

$$\text{precision} = (\#\text{ of true positives})/(\#\text{ of } |l_{ij}|, |u_{ij}| \geq 10^{-2}),$$
$$P(p) = \%\text{ test matrices with precision} \geq p,$$
$$SP(p) = \%\text{ of test matrices with soft precision} \geq p.$$



p	0.8	0.85	0.9
$P(p)$	86%	83%	80%
$SP(p)$	93%	91%	89%

Computing the Max-plus LU Factors

Let $\mathcal{A} \in \overline{\mathbb{R}}^{n\times n}$ have max-plus LU factors $\mathcal{L}, \mathcal{U}$ given by

$$\ell_{ik} = \mathrm{perm}\big(\mathcal{A}([1:k-1, i], 1:k)\big) - \mathrm{perm}\big(\mathcal{A}(1:k, 1:k)\big),$$
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Proposition (Hook, T.'16)

Let $G = (X, Y; E)$ be bipartite graph of $\mathcal{A}$ and M_ℓ be max weighted matching between $\{x(i)\}_{i=1}^{\ell}$ and $\{y(i)\}_{i=1}^{\ell}$.

- *u_{kj} is the weight of the maximally weighted path through the residual graph $R_G(M_{k-1})$ from $x(k)$ to $y(j)$ for $j \geq k$, or $-\infty$ if there is no such a path,*
- *ℓ_{ik} is the weight of the maximally weighted path through $R_G^T(M_k)$ from $x(k)$ to $x(i)$ for $i > k$, or $-\infty$ if there is no such a path.*

Max-plus ILU Preconditioner

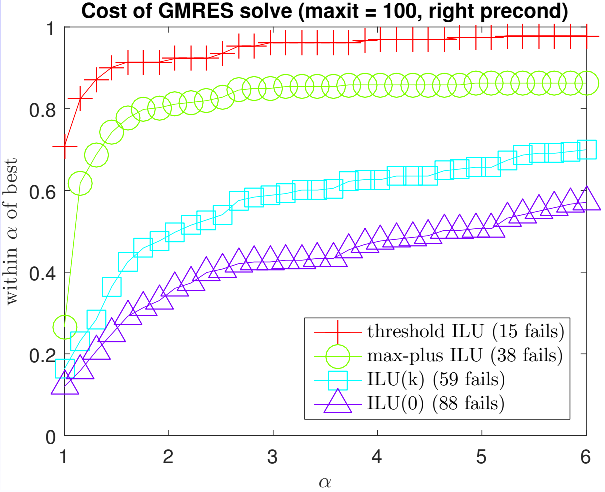
- Compute Hungarian scaling $H = PD_1AD_2$ of $A \in \mathbb{C}^{n\times n}$.
- Compute max-plus LU factors $\mathcal{L}$ and $\mathcal{U}$ of $\mathcal{V}_c(H)$.
- For a threshold t, define pattern matrix as

$$S_{ij} = \begin{cases} 1 & \text{if } \ell_{ij} \geq \log t \text{ or } u_{ij} \geq \log t, \\ 0 & \text{otherwise.} \end{cases}$$

- Compute ILU factors for H restricted to patter matrix S using, for example, the general static pattern ILU alg.

Performance Profile

233 matrices from U. of Florida sparse matrix collection.
Cost measure: # iters $\times\big((\text{nnz}(H) + \text{nnz}(L_{ilu}) + \text{nnz}(U_{ilu})\big)$.
Tolerance for GMRES: 10^{-5} (no restart).



Summary/Comments

- We presented a new method for approximating order of magnitude of entries in LU factors of a $A \in \mathbb{C}^{n\times n}$, which uses max-plus algebra and is based solely on $|a_{ij}|$.
- Cost: $O\big(n\tau + n^2 \log n\big)$. Can be parallelized.
- Approximation can be used to compute an ILU preconditioner for A.
- Max-plus ILU preconditioner tends to outperform ILU(k) and have performance very close to threshold ILU.

(see Hook, Tisseur, MIMS Eprint 2016.47)

- Can also define max-plus Cholesky factors and design incomplete Cholesky factorization preconditioners.

(see Hogg, Hook, Scott, Tisseur, MIMS Eprint 2016.59)

Polynomial Eigenvalue Problem (PEP)

Find $\lambda \in \mathbb{C} \cup \{\infty\}$ (eigenvalue) and nonzero $x, y \in \mathbb{C}^n$ (right/left eigenvectors) s.t.

$$P(\lambda)x = 0, \quad y^* P(\lambda) = 0.$$

where $P(\lambda) = \sum_{j=0}^{d} \lambda^j P_j$.

- Assume $P(\lambda)$ **regular**, i.e., $\det P(\lambda) \not\equiv 0$.
- P has dn eigenvalues. Finite eigenvalues are roots of $\det P(\lambda) = 0$.
- PEPs commonly solved by linearization:
  - converts P into a $nd \times nd$ linear pencil $A - \lambda B$,
  - solve generalized eigenvalue problem $(A - \lambda B)z = 0$, $w^*(A - \lambda B) = 0$,
  - recover e'vecs x, y of $P(\lambda)$ from those of $A - \lambda B$.

Tropical Scalar Polynomials (Max-Plus)

$$p(z) := \bigoplus_{j=0}^{d} p_j \otimes z^{\otimes j} = \max_{0 \le j \le d}(p_j + jz), \quad p_j \in \mathbb{R}_{\max}.$$

- $p(z)$ is a convex, piecewise-affine function.
- **Max-plus roots** are the points at which $p(z)$ is non-differentiable, i.e., points at which the maximum expression for $p(z)$ is attained by more than one term.
- $p(z)$ has d max-plus roots counting multiplicities, α_j, $j = 1, \dots, d$.
- $p(z) = p_d \otimes (\alpha_1 \oplus z) \otimes \cdots \otimes (\alpha_d \oplus z)$.

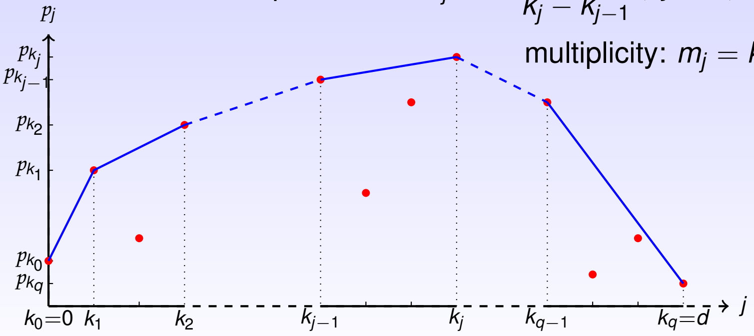
Tropical Scalar Polynomials (Max-Plus)

$$p(z) := \bigoplus_{j=0}^{d} p_j \otimes z^{\otimes j} = \max_{0\le j\le d}(p_j + jz), \quad p_j \in \overline{\mathbb{R}}.$$

Max-plus roots can be obtained via **Newton polygons** (upper convex hull of points (j, p_j), $j = 0\colon d$).

Tropical roots: $\alpha_j = -\dfrac{p_{k_j} - p_{k_{j-1}}}{k_j - k_{j-1}}$, $j = 1, \dots, q$,

multiplicity: $m_j = k_j - k_{j-1}$.



Scalar Polynomials: Classical/Max-Plus

"**Tropicalize**" $p(x) = \sum_{i=0}^{d} a_i x^i$, $a_i \in \mathbb{C}$, i.e., construct

$$\mathsf{p}(z) = \bigoplus_{i=0}^{d} \log|a_i| \otimes z^{\otimes i} = \max_{0 \le i \le d} (\log|a_i| + iz).$$

Let $\alpha_1 < \cdots < \alpha_q$ be roots of p with α_j of multiplicity m_j.

Theorem (Sharify'11)

If $\max(\alpha_j - \alpha_{j-1}, \alpha_{j+1} - \alpha_j) \ge \log 9 \approx 2.2$ *for* $1 \le j \le q$ *then* $p(x)$ *has exactly* m_j *roots in the annulus*

$$\mathcal{A}(x) = \{x \in \mathbb{C} : \tfrac{1}{3}\exp(\alpha_j) \le |x| \le 3\exp(\alpha_j)\}.$$

Max-plus roots of $\mathsf{p}(z)$ offer order of magnitude approx. to roots of p as long as the α_j are well separated.

Computation and Applications

Max-plus roots of max-plus polynomials

- can be computed in $O(d)$ **operations**, where $d = \deg(p)$,
- provide **asymptotic growth rates** of roots of $p(x;t) = \sum_{j=0}^{d} x^j \alpha_j(t)$;
- have been used for many years in **MPSolve** (Multiprecision Polynomial Solver) for the selection of the starting points in the Ehrlich-Aberth method. [Bini and Fiorentino, 2000]

Extension to Matrix Polynomials

Let $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{C}[\lambda]^{n\times n}$ and $p(x) = \bigoplus_{i=0}^{d} \log\|P_i\| \otimes x^{\otimes i}$

with max-plus roots $\alpha_1 < \cdots < \alpha_q$, α_j of multiplicity m_j.

$k_0 < \cdots < k_q$: corresponding indices in Newton polygon.

Theorem (Noferini, Sharify, T.' 14)

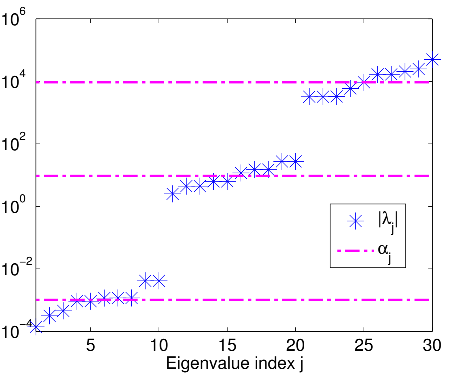
If $\alpha_\ell - \alpha_{\ell-1} \geq 2\log\big(1 + 2\kappa(P_{k_\ell})\big)$, $\ell = j-1, j$ then $P(\lambda)$ has exactly nm_j ei'vals inside the annulus $\mathcal{A}\Big((1+2\kappa(P_{k_{j-1}}))^{-1}\exp(\alpha_j), (1+2\kappa(P_{k_j}))\exp(\alpha_j)\Big)$.

For $P_{k_{j-1}}$, P_{k_j} well conditioned and $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$ sufficiently well separated, P has nm_j ei'vals of modulus close to $\exp(\alpha_j)$.

Here $\kappa(A) = \|A\|\|A^{-1}\|$.

Example: Random Cubic

```
n = 10;
A0 = randn(n); A1 = 1e3*randn(n);
A2 = 1e2*randn(n); A3 = 1e-2*randn(n);
```



Use of Max-Plus Roots in NLA

- Max-plus roots used to select **starting points** in the Ehrlich-Aberth method for polynomial eigenproblems. [Bini, Noferini, Sharify'13]

- Define **eigenvalue parameter scalings** ($\lambda = \exp(\alpha_j)\mu$) for polynomial eigensolvers based on linearizations.

$$\widetilde{P}(\mu) := \delta^{-1} P(\exp(\alpha_j)\mu), \quad \delta = \|P_{k_{j-1}}\| \exp(k_{j-1}\alpha_j).$$

  - Allow computation of ei'pairs with small b'err for $|\lambda|$ near $\exp(\alpha_j)$.
  - Linearization process does not affect ei'val condition number of ei'vals near $\exp(\alpha_j)$.
  - Available in quadratic eigensolver `quadeig`. [Hammarling, Munroe, Tisseur'13]

Lagrange Linearization

Rewrite $n \times n$ $P(\lambda) = \lambda^2 M + \lambda D + K$ in Lagrange basis,

$$P(\lambda) = \ell(\lambda)M + \beta_1 \ell_1(\lambda) P(\sigma_1) + \beta_2 \ell_1(\lambda) P(\sigma_2),$$

$\beta_j = (\sigma_j - \sigma_i)$, $\ell_j(\lambda) = \lambda - \sigma_i$, $i \neq j$, $\ell(\lambda) = \ell_1(\lambda)\ell_2(\lambda)$.

Construct $3n \times 3n$ pencil $A - \lambda B$, where

$$A = \begin{bmatrix} M & \beta_1 P(\sigma_1)/\sigma_1 & \beta_2 P(\sigma_2)/\sigma_2 \\ -I_n & -I_n & 0 \\ -I_n & 0 & -I_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-1}{\sigma_1} I_n & 0 \\ 0 & 0 & \frac{-1}{\sigma_2} I_n \end{bmatrix}.$$

- $L(\lambda)$ is a linearization of $\lambda^3 0_n + P(\lambda)$.
- Use tropical roots for interpolation points σ_j, $j = 1, 2$.
- Can show that $|\beta_j| \|P(\sigma_j)\| / |\sigma_j| = O(1)$, i.e., A is well-balanced.

Numerical Experiments

Backward error for eigenpair (λ, x) of $P(\lambda) = \sum_{j=0}^{d} \lambda^j P_j$,

$$\eta(\lambda, x) = \frac{\|P(\lambda)x\|_2}{(\sum_{j=0}^{d} |\lambda|^j \|P_j\|_2)\|x\|_2}.$$

Compare

- **Alg.1**: QZ alg applied to tropically scaled Lagrange linearization. [Van Barel, Tisseur'17]
- **Alg.2**: MATLAB's `polyeig` function.
- **Alg.3**: Gaubert & Sharify's algorithm. [Gaubert, Sharify'09]

Numerical Experiments (Cont.)

Largest backward errors of eigenpairs computed by **Alg.1–Alg.3**.

Problem	d	n	**Alg.1**	**Alg.2**	**Alg.3**
`cd_player`	2	60	4.1e-16	**3.1e-10**	**1.9e-13**
`damped_beam`	2	200	4.8e-16	**2.6e-11**	9.1e-17
`hospital`	2	24	3.9e-15	**2.9e-13**	2.6e-15
`orr_sommerfeld`	4	64	1.5e-15	**9.1e-08**	4.5e-15
`power_plant`	2	8	1.3e-16	**5.3e-12**	1.5e-18
`Problem 2`	7	4	8.5e-16	**1.2e-01**	**3.5e-12**
`Problem 17`	10	2	2.9e-15	**1.2e-01**	**7.7e-14**
`Problem 21`	5	4	7.5e-16	**2.7e-07**	**4.7e-10**

Conclusion

- Some NLA problems are easier to solve in the tropical algebra setting.
- Tropical analogues of NLA problems offer approximation to solutions of classical problems.
- These solutions are
  - usually cheap to compute,
  - usuful for the design of preprocessing steps and scalings.

Papers and tech reports available on my web page.