Elastic wave scattering from a cavity in a pre-stressed, nonlinear elastic medium

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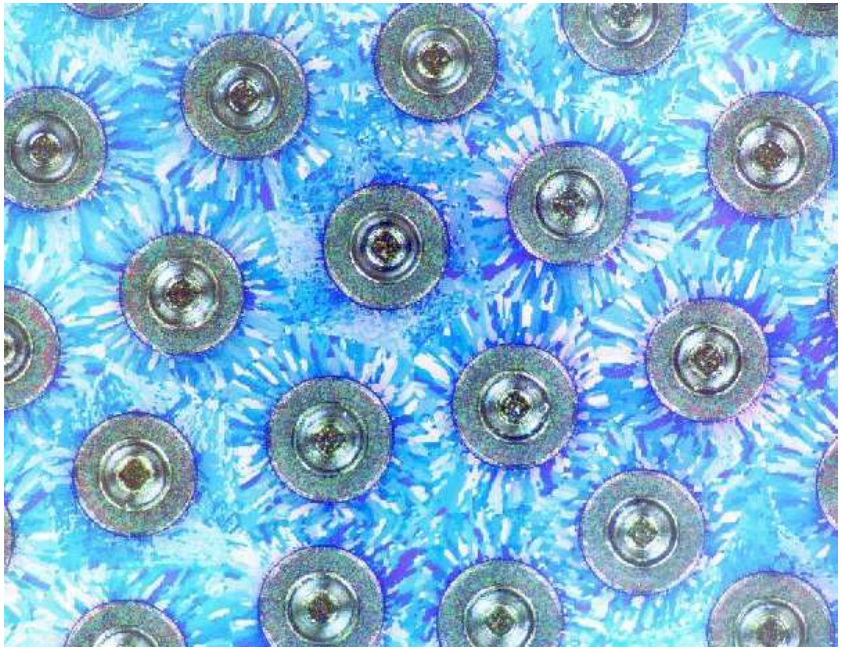
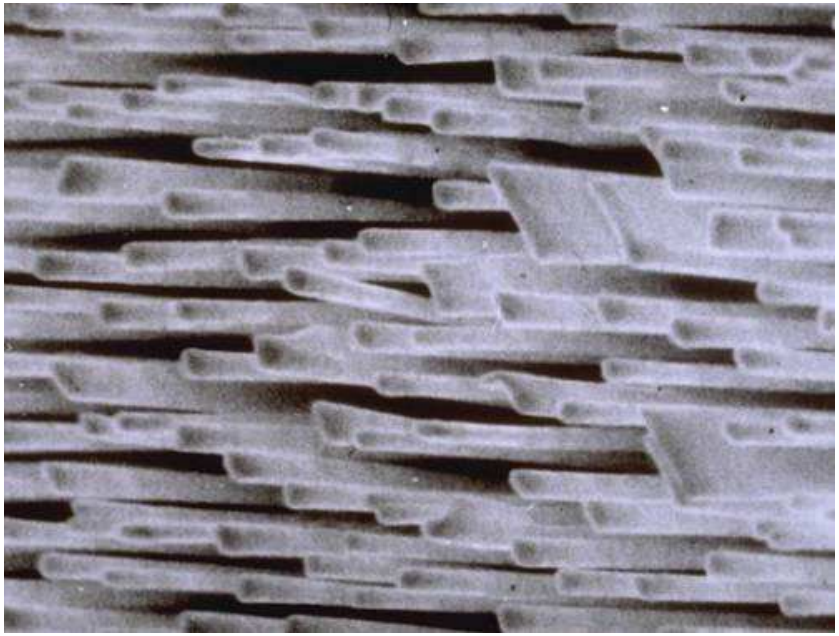
Composites





Overview

- The bigger picture
- Literature
- Applications

- A sub-problem: Scattering from a cavity in a pre-stressed nonlinear host medium
  - Hydrostatic pre-stress
  - Low frequency SH wave scattering
  - Matched asymptotic expansions
  - Effective shear modulus of a pre-stressed inhomogeneous medium

The bigger picture

Effective dynamic behaviour of inhomogeneous rubbery composites which have undergone pre-stress (small-on-large).

- The constitutive behaviour of rubber is difficult to describe
- Large deformation (geometric nonlinearity)
- Even simplified, nonlinear elastic behaviour (i.e. not Hookes Law), governed by $W = W(\lambda_1, \lambda_2, \lambda_3)$ and $\Sigma_j = \lambda_j / J dW/d\lambda_j$ is very difficult.
- Microstructure evolution (shape, position tracking, buckling, etc.)
- Changes in phase properties (density, elastic moduli, etc.)
- Rubber is strongly temperature and frequency dependent

Literature

Statics problem (effective nonlinear behaviour of a rubbery composite)

Hashin (1970's), Ogden (1970's)
Ponte-Castaneda and Willis (1990's/2000's)
The "rubber community" (numerous contributors) (1990's/2000's)

Influence of nonlinear pre-stress on subsequent linear waves

Pao, Sachse and Fukuoka, 1984, Physical Acoustics
Ogden, 1984, Nonlinear Elasticity
Destrade and Saccomandi, 2007 CISM course (prestressed media)

Interest has, however, almost exclusively centred on initial deformations which are homogeneous.

What about inhomogeneous deformations? Recently:

Parnell, W.J. 2007, IMA JAM
Bigoni D., Gei, M. and Movchan, A.B. 2008, JMPS
Bertoldi, K. and Boyce, M. 2008, Phys Rev E

In particular these authors showed the influence of pre-stress on the pass band structure in *periodic* solids.

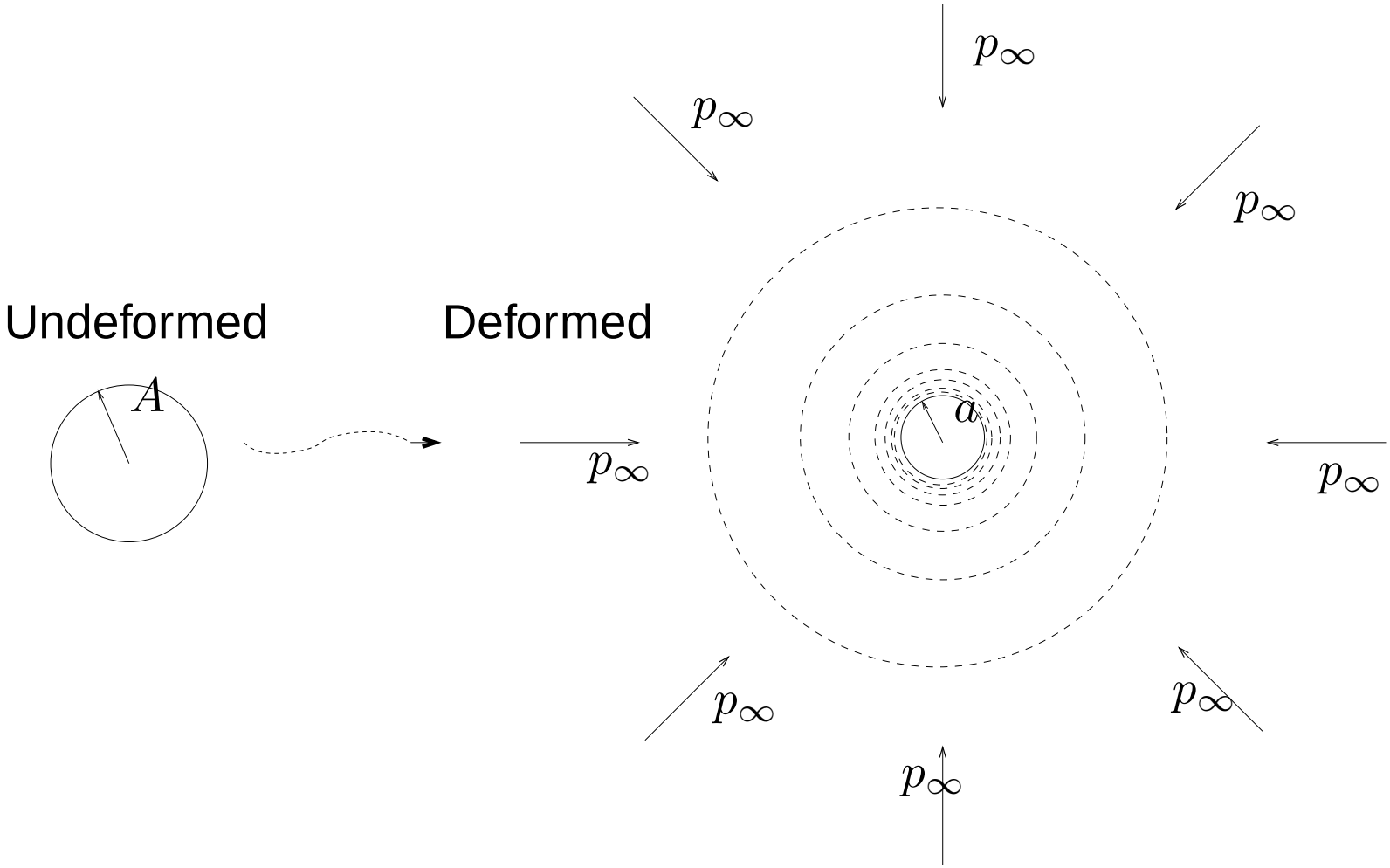
What about more general problems - canonical ones even?

Applications

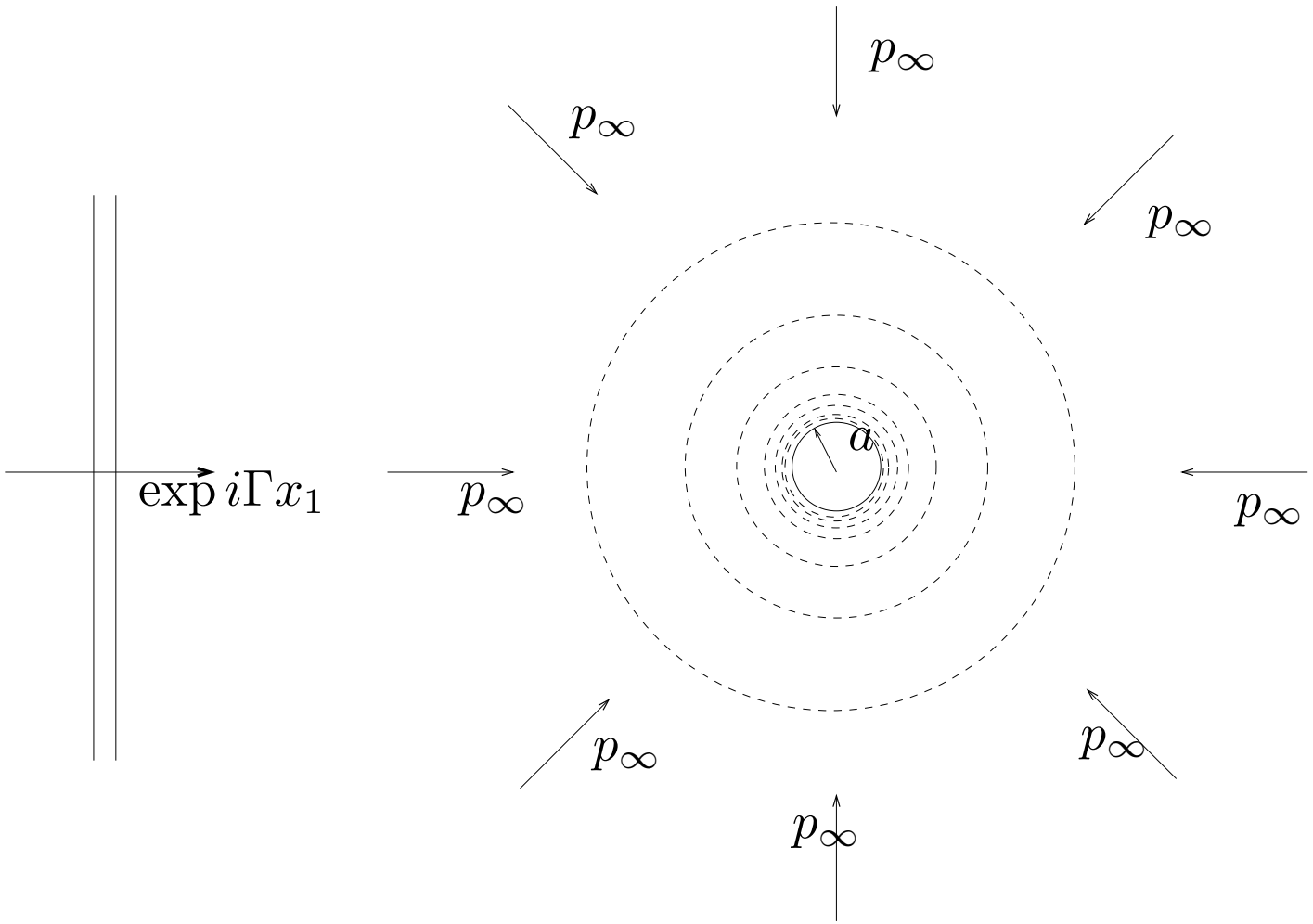
- Oil and geophysical industry
- NDT for pre-stressed materials
- Use of rubber composites in automotive, aerospace and defence industries (often in pre-stressed states)
- Biological tissues (lung, tendon, etc) are all nonlinear (visco)elastic, pre-stressed in vivo and are natural “composites”

Recent development - Quantum Tunnelling Composites - QTC’s

Problem description



Problem description



Wijewickerama and Leungvicharoen 2007:
"Scattering from a hole in a pre-stressed medium"
However, this relates only to homogeneous stretches around the hole.

No pre-stress

Non-dimensionalize lengthscales on reciprocal wavenumber $1/\Gamma$:

$$(\nabla^2 + 1)w = 0, \qquad \frac{\partial w}{\partial R} = 0 \text{ on } R = \epsilon_0 = A\Gamma$$

Incident plane wave:

$$w_{inc} = e^{iX_1} = J_0(R) + 2\sum_{n=1}^{\infty} i^n J_n(R)\cos n\Theta$$

Separation of variables gives

$$w_s = A_0 H_0(R) + 2\sum_{n=1}^{\infty} i^n A_n H_n(R)\cos n\Theta \text{ with } A_n = -\frac{J_n'(\epsilon_0)}{H_n'(\epsilon_0)}$$

and for $\epsilon_0 = A\Gamma \ll 1$,

$$A_0 = \frac{\pi}{4i}\epsilon_0^2 + o(\epsilon_0^2), \qquad A_1 = -\frac{\pi}{4i}\epsilon_0^2 + o(\epsilon_0^2)$$

Incompressibility and Constitutive behaviour

Assume host medium is incompressible.

Impose a hydrostatic pressure $\Sigma_{rr} = -p_\infty$ at infinity (far-field) and a fixed stretch L axially:

$$R = R(r) \qquad \Theta = \theta \qquad Z = \frac{z}{L}$$

Constitutive behaviour:

Take a Mooney-Rivlin medium:

$$W = \frac{\mu}{2}(S_1(I_1 - 3) + S_2(I_2 - 3)), \quad S_1, S_2 \in \mathbb{R}, \;\; S_1 + S_2 = 1.$$

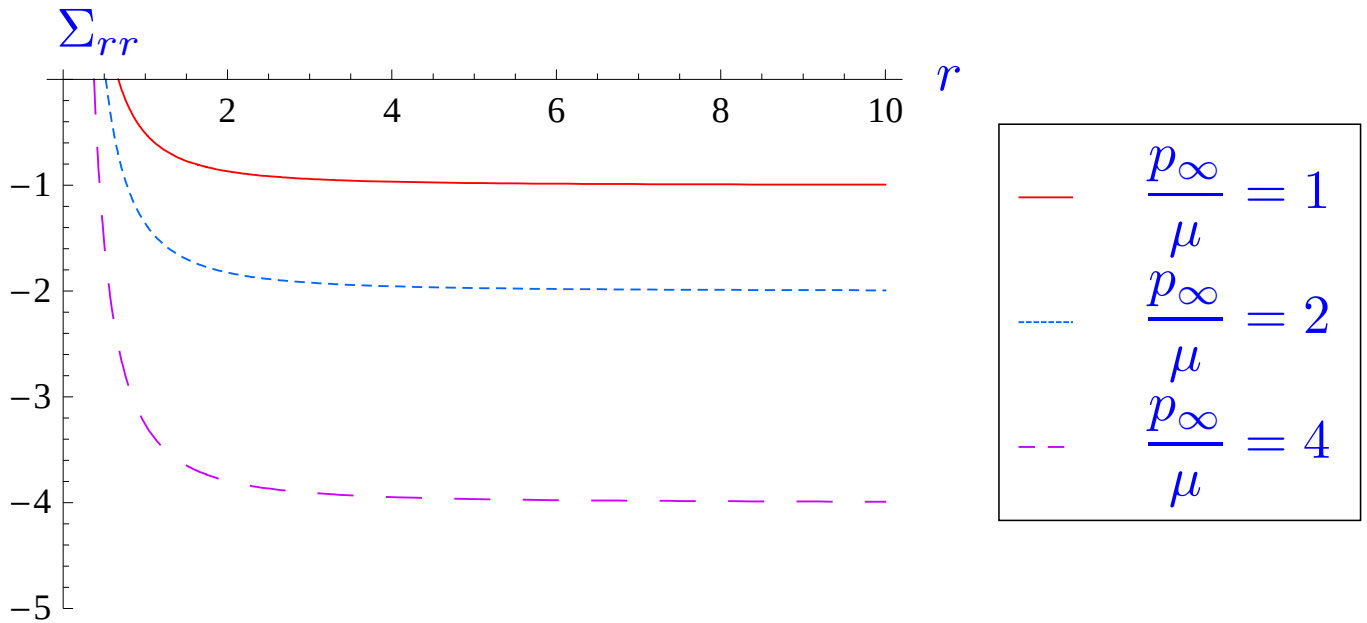
Stresses $\Sigma_{rr}, \Sigma_{\theta\theta}$ come from derivatives of this potential.

Radial stress, $L = 1$

Only 1 equilbrium equation:

$$\frac{d\Sigma_{rr}}{dr} + \frac{1}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) = 0$$

which can be integrated to give Σ_{rr}:

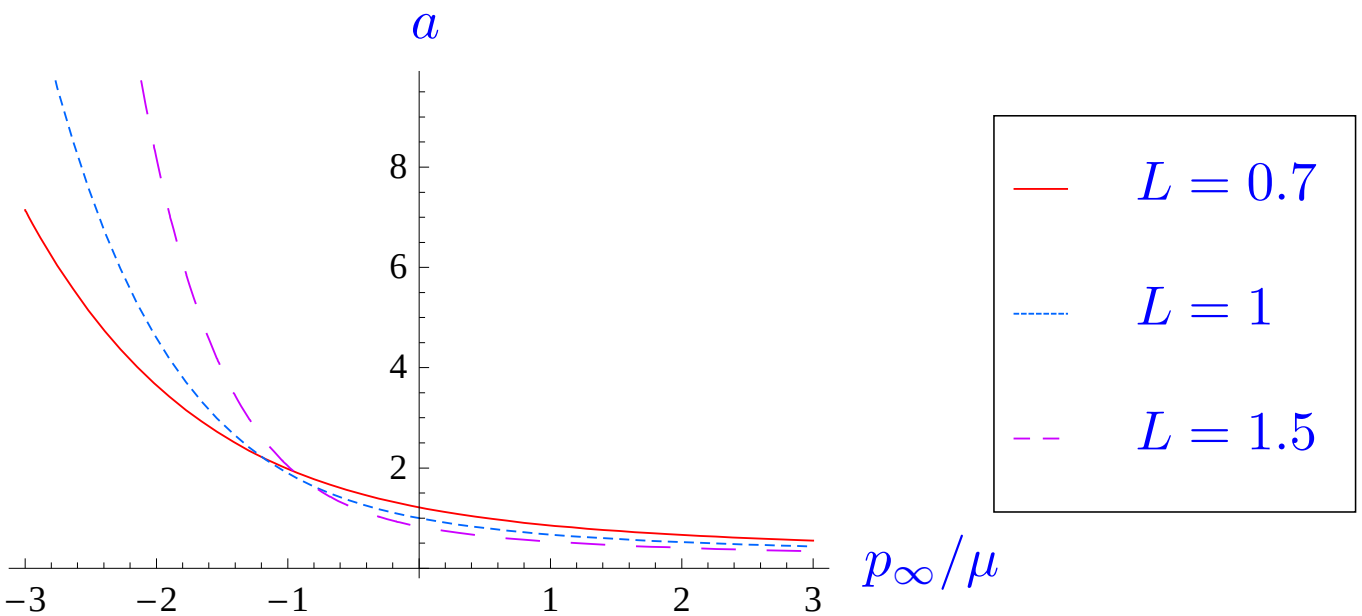


Deformed radius

Together with $\Sigma_{rr} \to -p_\infty$ as $r \to \infty$ this gives

$$\frac{p_\infty}{\mu} = -\frac{1}{2}\left(\frac{S_1}{L} + LS_2\right)\left(1 - \frac{A^2}{La^2} - \log\left(\frac{A^2}{La^2}\right)\right),$$

so that a can be determined in terms of $-p_\infty/\mu, L, A, S_1, S_2 = 1 - S_1$.



Incremental deformation

Superpose small amplitude waves on the finite deformation:

$$\hat{\mathbf{u}} = \mathbf{u} + \eta \mathbf{u}'$$

where $\mathbf{u}$ is the finite displacement, $\eta \ll 1$ and

$$\mathbf{u}' = (0, 0, w(r, \theta)) \exp(i\omega t).$$

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Transpires that the modified wave equation is:

$$\left(1 + \frac{k}{r^2}\right) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\left(1 - \frac{k}{r^2}\right) \frac{\partial w}{\partial r} + \frac{1}{r^2}\left(1 - \frac{k}{r^2 + K}\right) \frac{\partial^2 w}{\partial \theta^2} + \gamma^2 w = 0.$$

Can think of this as:

$$(\nabla^2 + \mathcal{L}_k + \gamma^2) w = 0$$

where $\mathcal{L}_k = 0$ for no pre-stress.

Low Frequency Scattering

Once again assume a low frequency plane incident wave incoming from infinity:

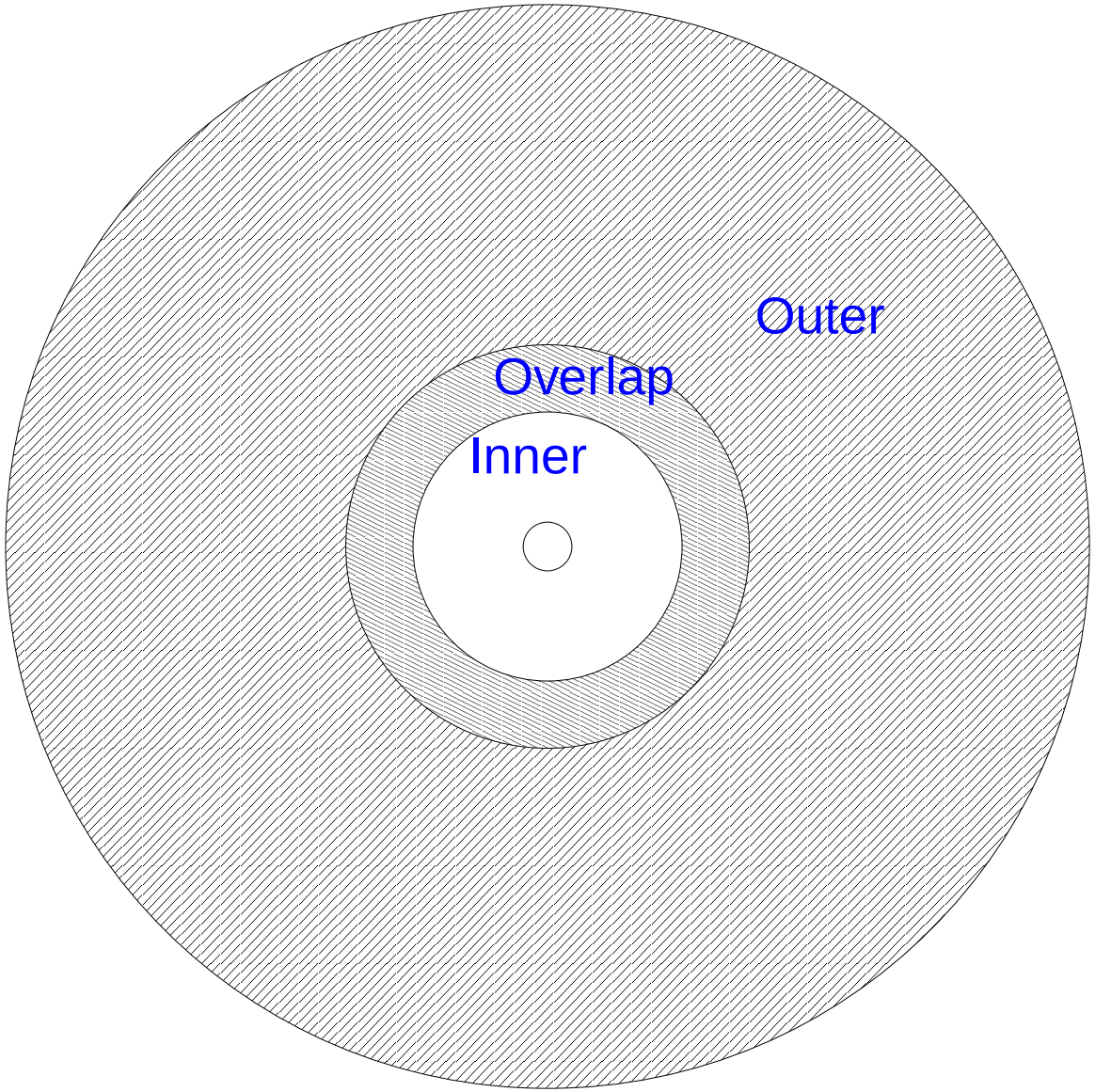
$$w_{inc} = e^{i\gamma r \cos\theta}$$

N.B. This does *not* satisfy the wave equation in the deformed medium (except at infinity), so the scattered wave is governed by:

$$(\nabla^2 + \mathcal{L}_k + \gamma^2)w_s + \mathcal{I}(r,\theta) = 0$$

Matched Asymptotics - Schematic

Scaling lengthscales on either a or γ gives inner and outer



Matched Asymptotics (Focus on Monopole)

Outer problem (scale r on γ)

Introducing $r_o = \gamma r$, we seek (separable) outer solutions of the form

$$w_s^{out} = F_0(r_o) + 2\sum_{n=1}^{\infty} i^n F_n(r_o)\cos n\theta$$

where

$$F_0(r_o) = \epsilon^2 F_0^{(2)}(r_o) + o(\epsilon^2),$$

This leading order term satisfies ($\hat{k} = k/a^2$):

$$\frac{d^2 F_0^{(2)}}{dr_o^2} + \frac{1}{r_o}\frac{dF_0^{(2)}}{dr_o} + F_0^{(2)} + \frac{\hat{k}}{r_o^2}\left(\frac{2}{r_o}J_1(r_o) - J_0(r_o)\right) = 0$$

Outer solution is:

$$F_0^{(2)}(r_o) = \hat{B}_0 H_0^{(1)}(r_o) + \mathcal{E}_0(r_o)$$

where $\hat{B}_0$ is a constant and $\mathcal{E}_0(r_0)$ is the inhomogeneous solution with limiting behaviour

$$\mathcal{E}_0(r_0) = -\frac{\hat{k}}{4} + O(r_0^2) \qquad r_0 \to 0$$

$$\mathcal{E}_0(r_0) = O\left(\frac{1}{r_0^{3/2}}\right) \qquad r_0 \to \infty$$

Inner Problem (scale r on a)

Full inner solution can also be written in separable form ($r_i = r/a$):

$$w_s^{in}(r_i) = f_0(r_i) + 2\sum_{n=1}^{\infty} i^n f_n(r_i)\cos n\theta$$

and the monopole term f_0 is written

$$f_0(r_i) = \epsilon^2 f_0^{(2)}(r_i) + o(\epsilon^2)$$

and satisfies

$$\left(1 + \frac{\hat{k}}{r_i^2}\right)\frac{d^2 f_0^{(2)}}{dr_i^2} + \frac{1}{r_i}\left(1 - \frac{\hat{k}}{r_i^2}\right)\frac{df_0^{(2)}}{dr_i} = 0$$

which has explicit solution (stress free on $r_i = \epsilon$)

$$f_0^{(2)}(r_i) = \frac{1}{4}(1+\hat{k})\log(r_i^2 + \hat{k}) + \hat{C}_0 \log\epsilon + D_0$$

Matching

Outer solution:

$$F_0(r_o) = \epsilon^2\left(\hat{B}_0 H_0^{(1)}(r_o) + \mathcal{E}_0(r_o)\right) + o(r_o^2)$$

Inner solution:

$$f_0(r_i) = \epsilon^2\left(\frac{1}{4}(1+\hat{k})\log(r_i^2+\hat{k}) + \hat{C}_0 \log\epsilon + D_0\right) + o(\epsilon^2)$$

Van Dyke's matching principle says that $f_0^{(1,1)} = F_0^{(1,1)}$ which gives the Modified Monopole Scattering Coefficient due to the pre-stress:

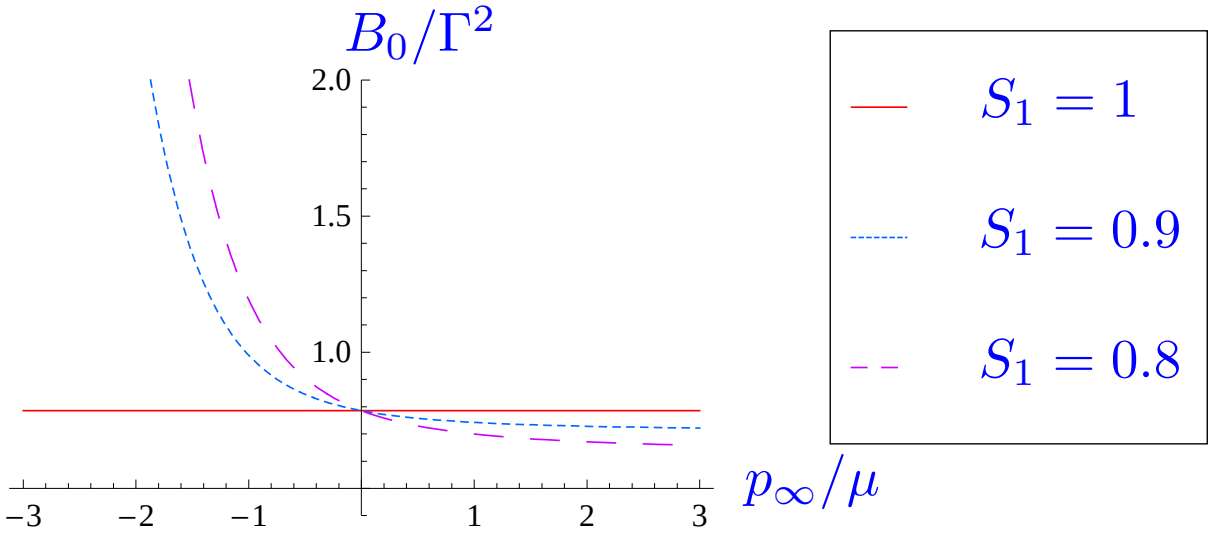
$$B_0 = \frac{\pi\epsilon_0^2}{4i}\left[\frac{a^2 L}{A^2(1+(L-1)S_1)}\left(1 + \frac{(A^2-a^2)LS_1}{a^2(1+(L-1)S_1)}\right)\right]$$

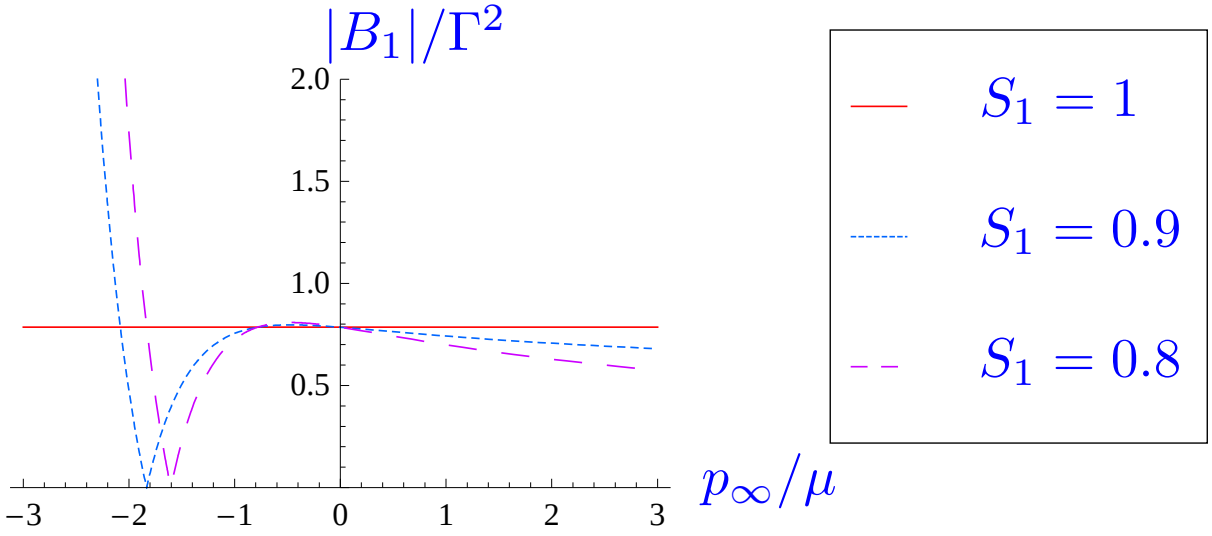
and perhaps a little surprisingly, when $S_1 = 1$ (Neo-Hookean)

$$B_0 = \frac{\pi\epsilon_0^2}{4i}.$$

Scattering Coefficients B_0 and B_1:

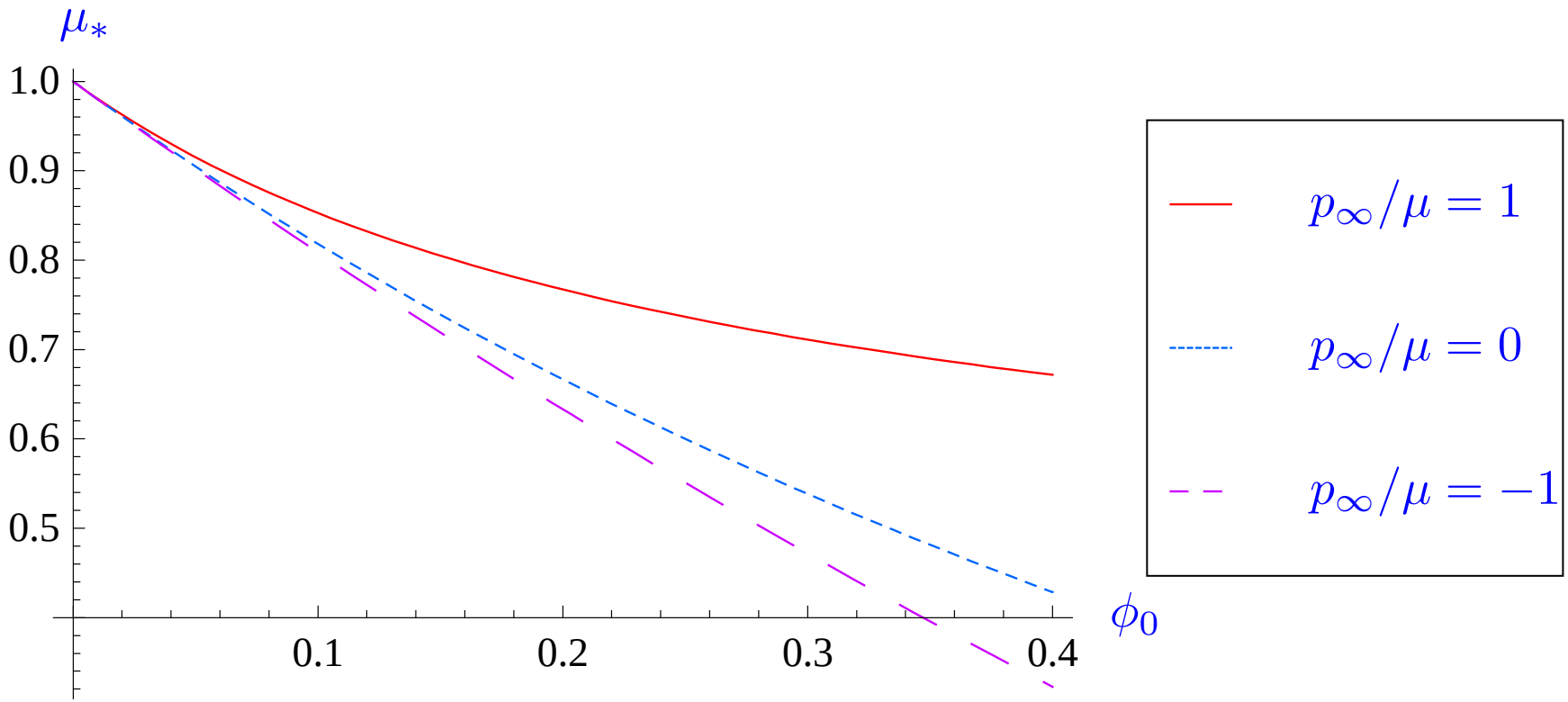
Set $L = 1$:





Effective shear properties

We can use this canonical problem in multiple scattering theory in order to deduce the effective response of a pre-stressed inhomogeneous medium:



Summary and acknowledgements

Summary

- Influence of inhomogeneous pre-stress on wave propagation
- In particular on low frequency scattering coefficients
- Impact on effective properties

Thanks to

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