## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction and motivation</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>Green’s functions in 1D</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>2.1 Ordinary Differential Equations: review</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>2.2 General forcing and the influence (Green’s) function</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>2.3 Linear differential operators</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>2.4 Sturm-Liouville (S-L) eigenvalue problems</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>2.5 Existence and uniqueness of BVPs for ODEs: The Fredholm Alternative</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>2.6 What is a Green’s function?</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>2.7 Green’s functions for Regular S-L problems via eigenfunction expansions</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>2.8 Green’s functions for Regular S-L problems using a direct approach</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>2.9 Green’s functions for the wave equation with time harmonic forcing</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>2.10 The adjoint Green’s function</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>2.11 Green’s functions for non S-A BVPs</td>
<td>49</td>
</tr>
<tr>
<td></td>
<td>2.12 Inhomogeneous boundary conditions</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>2.13 Existence of a zero eigenvalue - modified Green’s functions</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>2.14 Revision checklist</td>
<td>56</td>
</tr>
<tr>
<td>3</td>
<td>Green’s functions in 2 and 3D</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>3.1 Self-adjointness</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>3.2 An eigenvalue problem on a rectangular domain</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>3.3 Eigenvalue problem for the Laplacian operator</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>3.4 Multidimensional Dirac Delta Function</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>3.5 Green’s functions for the Laplace and Poisson equation</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>3.6 Applications of Poisson’s equation</td>
<td>69</td>
</tr>
<tr>
<td></td>
<td>3.7 Helmholtz’ equation in two spatial dimensions</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>3.8 Acoustic cloaking and metamaterials</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>3.9 Acoustic scattering by a cylinder and by arrays of cylinders</td>
<td>79</td>
</tr>
<tr>
<td>4</td>
<td>Theory of integral equations and some examples in 1D</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>4.1 Linear integral operators</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>4.2 What is an integral equation?</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>4.3 Volterra integral equations govern IVPs</td>
<td>84</td>
</tr>
<tr>
<td></td>
<td>4.4 Fredholm integral equations govern BVPs</td>
<td>86</td>
</tr>
</tbody>
</table>
### Application of Integral equations in 2 and 3D

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Distribution of heat source/electrical charge</td>
<td>90</td>
</tr>
<tr>
<td>5.2 An inclusion problem: Eshelby’s result</td>
<td>92</td>
</tr>
<tr>
<td>5.3 Homogenization for composite materials</td>
<td>98</td>
</tr>
<tr>
<td>5.4 Integral equations for wave scattering</td>
<td>98</td>
</tr>
</tbody>
</table>

### Curvilinear coordinate systems

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong> Curvilinear coordinate systems</td>
<td>99</td>
</tr>
</tbody>
</table>

### Useful stuff: Things that would be stated in the exam!

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1 Asymptotics of Bessel functions</td>
<td>101</td>
</tr>
<tr>
<td>B.2 Physical applications of the equations considered</td>
<td>101</td>
</tr>
</tbody>
</table>

### Example sheets

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C</strong> Example sheets</td>
<td>102</td>
</tr>
</tbody>
</table>
Syllabus

• **Section 1: Introduction and motivation.** What use are Green’s functions and integral equations? Some example applications. (0.5 lecture)

• **Section 2: Green’s functions in 1D.**
  Ordinary differential equations review, influence function, Linear differential operators, Green’s identity, adjoint and self-adjoint operators, Sturm-Liouville eigenvalue ODE problems, Fredholm Alternative, Green’s functions as eigenfunction expansions, dirac delta function and generalized functions, direct approach for determining Green’s functions via method of variation of parameters, the wave equation, adjoint Green’s function, non Sturm-Liouville problems, modified Green’s function and inhomogeneous boundary conditions. (8.5 lectures).

• **Section 3: Green’s functions in 2 and 3D.**
  Sturm-Liouville problems in 2 and 3D, Green’s identity, Multidimensional eigenvalue problems associated with the Laplacian operator and eigenfunction expansions, basics of Bessel functions, Green’s function for Laplace’s equation in 2 and 3D (unbounded and simple bounded domains) and associated applications, Green’s function for Helmholtz equation in 2D (unbounded and simple bounded domains) and associated wave scattering and cloaking problems. (5 lectures).

• **Section 4: Integral equations in 1D.**
  Linear integral operators and integral equations in 1D, Volterra integral equations govern initial value problems, Fredholm integral equations govern boundary value problems, separable (degenerate) kernels, Neumann series solutions and iterated kernels, applications to scattering. (4 lectures)

• **Section 5: Integral equations in 2 and 3D.**
  Integral equations associated with Laplace’s/Poisson’s equation and applications in potential flow, electromagnetism and thermal problems. Eshelby’s conjecture for heat conduction. Applications to homogenization and effective material properties. Formulation of integral equation for scalar wave scattering in 2D. (4 lectures)
Course lecturer

The course lecturer is Dr. William Parnell (william.parnell@manchester.ac.uk). My office is 2.238 in the School of Mathematics, Alan Turing building. If you have any questions please use either email or preferably ask me questions directly after the lectures. You will have plenty of time to discuss further aspects in the examples classes.

Course arrangements

There will be two lectures per week in weeks 1-11 and one examples class per week in weeks 2-12. 12 example sheets will be set, with sheet \( n \) being worked on in week \( n \), \( n = 1, 2, 3, ..., 12 \). Examples classes will be held in weeks 2-12. There is no class in week 1 as Example sheet 1 is revision of material you should know. If you cannot do it, look back at your MT10121 and MT20401 notes but of course ask me if in the end you are still having problems. Students should work on the examples sheets before the Example class so that they can flag up any difficulties. Some hours in week 12 will be set aside for revision as should be expected.

Lectures are held on Mondays, 11.00-11.50 in the Schuster Moseley Lecture theatre and Fridays 13.00-13.50 in Alan Turing, G.107. The examples class follows the Friday class, 14.00-14.50 also in G.107. The purpose of this is for you to work through some of the problems on the examples sheet that you have already looked at and ask for help if you need it.

The end of semester 2 hour examination accounts for 80% and a mid-term 50 minute test on the Friday of week 7 accounts for 20%. The test will be on material from Section 2 only and the accompanying Example sheets 1-5 of the course. This test will help you with revision and it is good to get it out of the way before Easter.

A note about the notes

These notes are pretty comprehensive. You should not really need to look at any other books as a result of this. You may also have to look back at your notes from MT10121 and MT20401 from time to time. There are plenty of examples provided both in the notes and on the Examples sheets. In the lectures I will go through most of the notes but not always all of the details. The notes accompany the lectures and you should certainly still attend and listen carefully even though I provided these notes. I certainly will not necessarily write all of the text on the board although I will mention and describe all of the related mathematical ideas. It is up to you to read the notes carefully. In lectures I will mainly focus on the mathematics, the theory and model examples to aid understanding. Sometimes I will ask you to work through some of the examples in the notes in your own time. And remember you need to spend your own time reading through the notes to understand them!

You will notice that at the end of each section I provide a revision check-list. This should help you to understand what you do and do not understand at the end of the section with the aid of the notes and the related examples sheets.

I urge you to look at the examples sheets before the examples class. Otherwise you will not make the most of the help available in the session and you may fall behind.
1 Introduction and motivation

In this course and these notes we will discuss the solution to a broad class of problems in applied mathematics. We will largely focus on solving ordinary differential equations (ODEs) and partial differential equations (PDEs). These will take the form

\[ \mathcal{L}u(x) = f(x) \]

for ODEs and we are interested in boundary value problems where \( x \in [a, b] \) for some real \( a \) and \( b \) with boundary conditions prescribed on \( a \) and \( b \). In the end we want to solve for the field variable \( u(x) \). We can also analyse initial value problems where initial conditions are specified at \( x = 0 \) but we only have 22 lectures! Here \( \mathcal{L} \) is known as an ordinary differential operator, e.g. \( \mathcal{L} = d^2/dx^2 \). The function \( f(x) \) is a “forcing” function. PDEs will take the form

\[ \mathcal{L}u = Q(x) \]  \( (1.1) \)

on some domain \( x \in D \) where \( x = (x, y, z) \) in three dimensional problems. Here \( \mathcal{L} \) is a partial differential operator, e.g. \( \mathcal{L} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \), the Laplacian. Note that for reasons of clarity and time restrictions we do not consider problems with explicit time dependence or forcing, or rather we consider certain types of time dependent problems, e.g. \( \exp(-i\omega t) \) for wave problems\(^1\) which yield problems in the form (1.1) where (1.1) is the steady-state forcing. We also restrict attention to scalar problems so that \( u \) is a scalar field (temperature, pressure, etc.). As a result of the above, the PDEs that we consider in this course are all elliptic. This therefore includes the steady state heat equation (Laplace’s equation \( \nabla^2 u = Q(x) \)) and the time harmonic wave equation (Helmholtz equation \( \nabla^2 u + k^2 u = Q(x) \)). “We’ve done all this before” you may say. Well you have done some of it, but we will be learning about a special technique to solve inhomogeneous PDEs, i.e. when the forcing terms \( f(x) \) and \( Q(x) \) above are non-zero. This technique is the method of Green’s functions\(^2\). It transpires that the solution to the problem can (in general) be written as a weighted integral of the forcing over the domain, where the weighting is the Green’s function. This is a topic that has been and is still of great interest as a research topic in applied mathematics. Green’s functions have pervaded many areas of mathematics, science, engineering and computation, often in surprising ways. In particular, Green’s functions can be used in order to re-write the differential equation forms of the problems in integral equation form. The subject of boundary element methods, an area of great interest for solving problems numerically, stems from this development.

In addition to the fact that they are of great use, they are also very interesting mathematically. We will be able to discuss various ideas and theoretical aspects pertaining to the theory of ordinary and partial differential equations.

As an example of the use of Green’s functions, consider the simple ordinary differential equation of the form

\[ \frac{d^2u}{dx^2} = f(x) \]  \( (1.2) \)

\(^1\)In some of the Example Sheets we do consider a small subset of time dependent PDEs. The reason for doing this is to see the context in which our problems without time dependence reside.

\(^2\)named after the brilliant applied mathematician and Nottingham Miller George Green (1793-1841) who developed them as a tool in the 1830s.
where $f$ is some forcing function, on a domain $x \in [0, L]$ with homogeneous boundary conditions e.g. $u(0) = 0, du/dx(L) = 0$. This corresponds to the steady state heat equation in one dimension with heat source term $f(x)$ and with fixed temperature at $x = 0$ and an insulated boundary at $x = L$ (no heat flux across the boundary). This problem is of course a boundary value problem, i.e. an ODE governing some function $u$ (the temperature) with corresponding boundary conditions at the edge of the domain.

It transpires that a solution of the problem can be written in the form

$$ u(x) = \int_{0}^{L} G(x, x_0) f(x_0) \, dx_0 \tag{1.3} $$

where $G(x, x_0)$ is the corresponding Green’s function which satisfies an associated boundary value problem. We will not describe this here but will of course in detail in later chapters. Note that (1.3) is strictly an integral equation, although it does not have to be solved so it can be said to be an integral expression for the function $u(x)$.

In two and three dimensions, the corresponding solution can be written

$$ u(x) = \int_{V} G(x, x_0) f(x_0) \, dx_0 \tag{1.4} $$

where $D$ is the two/three dimensional domain and $G$ is the corresponding Green’s function.

We will describe the theory behind the above analysis and describe in particular some applications in the context of heat conduction and wave propagation. In particular for problems involving inhomogeneous media (think of a solid body with an “inclusion” embedded inside it) we are able to write down integral equations which govern the scalar field $u(x)$. We shall describe methods to solve these interesting problems. Indeed in later chapters we will make links to some modern research topics. These include “acoustic scattering theory” i.e. how sound waves are scattered from obstacles, “acoustic cloaking theory” i.e. how we can try to make objects “invisible” to sound and the study of “composite materials”, although we probably will not have the time to consider all of these applications. I will of course make it clear what is and is not examinable.

Here are some brief details of the application areas described above.

**Acoustic scattering theory**

Suppose that we have a uniform medium and within this domain we embed an “inclusion”, it could have arbitrary shape. Imagine that sound (acoustic) waves are incident on the inclusion. This causes the waves to be scattered. How do we solve for this scattered field? One example is shown in figure 1. We shall describe how we do this for simple geometries in section 3 via Green’s functions. In section 5 we describe a more general case and describe how the problem can be reformulated in terms of integral equations. We describe a technique that can be implemented in order to predict the scattered field.

---

3In harder problems this not the case - we will consider some of these in sections 4 and (5) when we discuss integral equations.
Figure 1: An acoustic (sound) field is generated by “forcing” at the point in the white circle. Outgoing circular waves are generated. These outgoing waves are subsequently scattered by the circular black region. Because in this instance the wavelength is commensurate with the size of the circular region, scattering is strong: we see a clear shadow region and backscattered field. The field is time harmonic so that we are showing the amplitude of the wave field at a single instant in time.

Acoustic cloaking theory

Suppose that we did not want the field to be scattered from the circular region above. How could we enable this to happen? The development of the two and three dimensional Green’s function enables us to easily describe the concept of acoustic cloaking. This is a topic of great interest presently. The idea is to design an acoustic material which possesses properties in order to “guide” the acoustic waves around a region of interest. See figure 2. This is of interest in a number of applications mainly due to the fact that outside the cloak region, one cannot tell at all that there is a circular region or anything inside it. We will describe how this concept of cloaking can be achieved theoretically in section 3.

Composite materials

Suppose that we have a material which consists of lots of small inclusions embedded inside an otherwise uniform “host” medium (see figure 3). This type of so-called composite material is used in thousands of applications in engineering, medical science, the automotive and defence industries and aerospace sector amongst many others. If the inclusions and host medium have different thermal conductivities, how do we theoretically predict
Figure 2: A material with special material properties is wrapped around the black circular region. These properties guide the incoming acoustic (sound) wave, generated at the “point” just to the right of the image, around the region. The region is therefore cloaked and anything inside will not be “seen” in the far-field. The field is time harmonic so that we are showing the amplitude of the wave field at a single instant in time.

what the so-called overall (or effective) thermal conductivity is and how it depends on the volume fraction (relative quantities of the different constituents), conductivities and shape of the constituents of the material in question? In section 5 we will use integral equations in order to motivate one approach to solving this problem. It transpires that we can introduce a small amount of the inclusion material in order to significantly influence (and improve) the overall (or effective) thermal conductivity of the material. This can assist in decreasing the cost, improving the effectiveness, etc. of the material.

Interesting mathematics underlies these applications!

The three applications above will be considered in this course but note that above all we will be interested in the interesting mathematics that sits underneath and describes these important phenomena. Understanding the mathematics is key to getting sensible predictions in these application areas. These research areas are of great current interest and many scientists are currently undertaking related mathematical research with associated applications in physics, materials science, chemistry, medical imaging and diagnostics, medical implants, non destructive evaluation of components in industry and many more.
We show a composite material which consists of many small inclusions distributed throughout a uniform “host” material. The question is how do we predict the overall material properties from knowledge of the constituent materials?
2 Green’s functions in 1D

We now come on to the introduction of the concept of a Green’s function and we shall start in one dimension, i.e. with *ordinary differential equations* (ODEs). We will usually be interested in solutions of *second order* (highest derivative is two) ODEs. This includes many problems that are of interest in practice, for example the (steady state) heat equation and the wave equation at fixed frequency.

### 2.1 Ordinary Differential Equations: review

You have seen the material here before (MT10121). We will review it briefly but look back at your notes to ensure that you know it thoroughly!

Let us consider second order Ordinary Differential Equations (ODEs) of the form

\[ p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x) \]  

where \( p(x), r(x), q(x) \) and \( f(x) \) are real functions. Two type of problems can be considered: *Boundary Value Problems* (BVPs) and *Initial Value Problems*. For BVPs, \( x \) is a spatial variable e.g. \( x \in [a, b] \) and we require associated boundary conditions (BCs) e.g. \( \mathcal{B} = \{u(a) = 0, u(b) = 1\} \), etc. For IVPs, \( x \) is time so \( x \in [0, \infty) \) and we require associated initial conditions (ICs) e.g. \( \mathcal{I} = \{u(0) = 0, u'(0) = 1\} \). If the BCs or ICs have a zero right hand side they are known as homogeneous. Otherwise they are known as inhomogeneous. We will consider exclusively BVPs in this section. We will consider IVPs in section 4 (integral equations in 1D).

Note that often we can divide through by \( p(x) \) in order to give a unit coefficient of \( u''(x) \). However in general we have to be careful with this. Some singular problems (that are physical) do not allow us to do this.

The *general solution* of the ODE is in general written in the form

\[ u(x) = u_c(x) + u_p(x) \]  

where \( u_c(x) \) is known as the complementary function and is the solution to the homogeneous ODE

\[ p(x)u_c''(x) + r(x)u_c'(x) + q(x)u_c(x) = 0 \]  

whereas \( u_p(x) \) is known as the particular solution and is the solution to the inhomogeneous ODE

\[ p(x)u_p''(x) + r(x)u_p'(x) + q(x)u_p(x) = f(x). \]  

Once we have determined (2.2) it will have some undetermined constants (these are always in the complementary function) which are then determined by *imposing the BCs or ICs on the general solution*.

How do we determine the complementary function and particular solution? Let us discuss this now. We note that in particular we are interested in two types of ODEs: Constant coefficient ODEs and those of Euler type since these may be solved analytically. ODEs that cannot be solved analytically can of course be treated by numerical methods but this is outside the scope of this course.
2.1.1 Homogeneous ODEs: The complementary function

For constant coefficient ODES, with \( r, q \in \mathbb{C} \) we can write

\[
u''_c(x) + ru'_c(x) + qu_c(x) = 0. \quad (2.5)
\]

Here we really can take the coefficient of \( u''_c(x) \) to be unity since we can divide through by the constant \( p \). We know that since the ODE is second order there will be two fundamental solutions say \( u_1(x) \) and \( u_2(x) \) that contribute to the complementary function and it can be written as \( u_c(x) = c_1u_1(x) + c_2u_2(x) \) for some real constants \( c_1, c_2 \in \mathbb{R} \). To find \( u_1 \) and \( u_2 \), seek solutions of the form \( \exp(mx) \) where \( m \in \mathbb{R} \) and find the \( \lambda \) that ensure solutions from \( m^2 + rm + q = 0 \). There will either be two real, two complex conjugate or repeated roots. In the case of the latter one of these solutions must be multiplied by \( x \) in order to obtain the second linearly independent solution (see question 4 of Example sheet 1).

**Example 2.1** Find the solution of

\[
u''_c(x) + u'_c(x) - 2u_c(x) = 0. \quad (2.6)
\]

*Seeking solutions in the form \( \exp(mx) \) gives\( m^2 + m - 2 = (m + 2)(m - 1) = 0 \) so that \( m = -2, 1 \). The solution is therefore \( u_c(x) = c_1 \exp(-2x) + c_2 \exp(x) \) (2.7)*

for some constants \( c_1, c_2 \).

**Example 2.2** Find the solution of

\[
u''_c(x) + 2u'_c(x) + u_c(x) = 0. \quad (2.8)
\]

*Seeking solutions in the form \( \exp(mx) \) gives \( m^2 + 2m + 1 = (m + 1)^2 = 0 \) so that \( \lambda = -1 \) (repeated). The solution is therefore \( u_c(x) = c_1 \exp(-x) + c_2x \exp(-x) \) (2.9)*

for some constants \( c_1, c_2 \).

Euler equations are of the form

\[
x^2u''_c(x) + rxu'_c(x) + qu_c(x) = 0 \quad (2.10)
\]

for some \( r, q \in \mathbb{R} \) and \( x \neq 0 \). Solutions are then sought in the form \( x^m \).

**Example 2.3** Find the solution of the Euler ODE

\[
x^2u''_c(x) + 2rxu'_c(x) - 6u_c(x) = 0. \quad (2.11)
\]

*Seeking solutions in the form \( x^m \) gives \( m(m - 1) + 2m - 6 = m^2 + m - 6 = (m + 3)(m - 2) \) so that \( m = 2 \) and \( m = -3 \). The solution is therefore \( u_c(x) = c_1x^2 + \frac{c_2}{x^3} \) (2.12)*

for some constants \( c_1, c_2 \).
2.1.2 Inhomogeneous ODEs

Let us now consider how we find the particular solution \( u_p(x) \). We can obtain this by two alternative techniques: the method of undetermined coefficients and the method of variation of parameters.

**Inhomogeneous ODEs: Method of undetermined coefficients**

Consider again the general second-order ODE of the form

\[
p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x).
\]

We must seek particular solutions \( u_p(x) \) in order to take care of the inhomogeneous term \( f(x) \) on the right hand side. A simple method is known as the method of undetermined coefficients. This is sometimes also called the method of intelligent guessing!

**Example 2.4** Find the particular solution for the ODE

\[
u''(x) + u'(x) - 2u(x) = 10 \exp(3x)
\]

We note that \( \exp(3x) \) is not one of the fundamental solutions (you can check this). Therefore pose a particular solution in the form \( u_p(x) = a \exp(3x) \) for some \( a \in \mathbb{R} \) to be determined. Substituting this into the ODE we find that

\[
a(9 \exp(3x) + 3 \exp(3x) - 2 \exp(3x)) = 10 \exp(3x)
\]

and so for consistency we note that we require \( a = 1 \).

If the right hand side of the ODE is one of the fundamental solutions we multiply our choice by \( x \) (note the special case of an Euler ODE with fundamental solution \( 1/x \) with forcing term \( 1/x \) would have \( u_p(x) = (a/x) \ln x \)). Clearly this method can sometimes be difficult to apply because we are using our judgement as to what we should choose as a candidate solution. It would be preferable if we could derive a more algorithmic approach.

**Inhomogeneous ODEs: Method of variation of parameters**

We cannot always use the method of undetermined coefficients. Sometimes we just cannot “see” the particular solution. Consider again the general second-order ODE of the form

\[
p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x).
\]

We will now briefly describe the method of variation of parameters. In order to apply this method we need to know the complementary function. This is imperative (remember that this was not the case with the method of undetermined coefficients). We know from section 2.1.1 that the complementary function has the form

\[
u_c(x) = c_1 u_1(x) + c_2 u_2(x).
\]
We will pose a particular solution of the form
\[ u_p(x) = v_1(x)u_1(x) + v_2(x)u_2(x) \]  \hspace{1cm} (2.18)
and so we need to determine the two unknown functions \( v_1(x) \) and \( v_2(x) \).

Let us differentiate \( u_p(x) \):
\[ u'_p(x) = v'_1(x)u_1(x) + v_1(x)u'_1(x) + v'_2(x)u_2(x) + v_2(x)u'_2(x) \]  \hspace{1cm} (2.19)
and make the assumption that
\[ v'_1(x)u_1(x) + v'_2(x)u_2(x) = 0. \]  \hspace{1cm} (2.20)
Differentiate \( u''_p(x) \) again
\[ u''_p(x) = v''_1(x)u_1(x) + v_1''(x)u_2(x) + v_2''(x)u_1(x) + v_2(x)u''_2(x). \]  \hspace{1cm} (2.21)
Substituting \( u_p(x) \) and its derivatives into the governing ODE and rearranging we find
\[
p(x)[v'_1(x)u'_1(x) + v'_2(x)u'_2(x)] + v_1(x)[p(x)u''_1(x) + r(x)u'_1(x) + q(x)u_1(x)] + v_2(x)[p(x)u''_2(x) + r(x)u'_2(x) + q(x)u_2(x)] = f(x). \]  \hspace{1cm} (2.22)
Of course in the second and third terms on the left hand side, the terms in square brackets are zero. Therefore
\[
p(x)(v'_1(x)u'_1(x) + v'_2(x)u'_2(x)) = f(x). \]  \hspace{1cm} (2.23)
This together with the assumption (2.20) gives us two equations to solve for \( v'_1(x) \) and \( v'_2(x) \). We solve to find
\[
v'_1(x) = \frac{-u_2(x)f(x)}{p(x)(u_1(x)u'_2(x) - u_2(x)u'_1(x))}, \quad v'_2(x) = \frac{u_1(x)f(x)}{p(x)(u_1(x)u'_2(x) - u_2(x)u'_1(x))}. \]  \hspace{1cm} (2.24)
We note that since \( u_1(x) \) and \( u_2(x) \) are fundamental solutions the Wronskian is non-zero:
\[ W(x) = u_1(x)u'_2(x) - u_2(x)u'_1(x) \neq 0. \]  \hspace{1cm} (2.25)
So, we can integrate in each of (2.24) between \( a \) and \( x \) to find
\[
v_1(x) = \int_a^x \frac{-u_2(x)f(x)}{p(x)W(x)} \, dx_0 + v_1(a), \quad v_2(x) = \int_a^x \frac{u_1(x)f(x)}{p(x)W(x)} \, dx_0 + v_2(a). \]  \hspace{1cm} (2.26)
We can set \( v_1(a) = v_2(a) = 0 \) because from (2.18) these merely generate additional terms that are of the form of the complementary function. Therefore
\[
v_1(x) = \int_a^x \frac{-u_2(x)f(x)}{p(x)W(x)} \, dx_0, \quad v_2(x) = \int_a^x \frac{u_1(x)f(x)}{p(x)W(x)} \, dx_0. \]  \hspace{1cm} (2.27)
Therefore we can assert that the general solution to the ODE is
\[
u(x) = u_c(x) + u_p(x) = (c_1 + v_1(x))u_1(x) + (c_2 + v_2(x))u_2(x) \]  \hspace{1cm} (2.28)
2.2 General forcing and the influence (Green’s) function

In order to give a full description of Green’s functions, what they are and why they are useful we need a lot more ODE theory some (most?) of which you will not have come across before. We will come on to this in a moment but let us consider a simple problem here first in order to motivate the idea of a Green’s function.

In particular we should ask if we can obtain a solution form for an ODE with an arbitrary forcing term \( f(x) \) on the right hand side? In order to answer this question let us consider a canonical problem and one that has a very important application. Consider the simple equation

\[
d^2 u / dx^2 = u''(x) = f(x)
\]  

(2.30)

on the domain \( x \in [0, L] \) subject to homogeneous boundary conditions \( B = \{ u(0) = 0, u(L) = 0 \} \). This problem is in fact the steady state heat equation. I.e. the heat equation without any time dependence\(^4\). Temperature is fixed to be zero on the boundaries.

In order to solve this problem, we note that the complementary function satisfies

\[
u''(x) = 0
\]  

(2.31)

and by direct integration, the fundamental solutions are 1 and \( x \). However it turns out to be very convenient to have fundamental solutions one of which satisfies one of the homogeneous boundary conditions and one of which satisfies the other. Therefore we choose linear combinations, to obtain

\[
u_1(x) = x, \quad u_2(x) = L - x
\]  

(2.32)

satisfying the left and right boundary condition respectively.

Using (2.27), since \( W = u_1u_2' - u_2u_1' = x(-1) - (L - x)(1) = -L \), we find that

\[
v_1(x) = \frac{1}{L} \int_0^x f(x_0)(L - x_0) \, dx_0
\]  

(2.33)

\[
v_2(x) = -\frac{1}{L} \int_0^x f(x_0)x_0 \, dx_0
\]  

(2.34)

The full solution is therefore

\[
u(x) = (c_1 + v_1(x))x + (c_2 + v_2(x))(L - x)
\]  

(2.35)

so finally let us apply the BCs. Setting \( x = 0 \) means that \( c_2 = 0 \) and for \( x = L \) we find

\[
0 = (c_1 + v_1(L))L
\]  

(2.36)

so that \( c_1 = -v_1(L) \). We then note that

\[
c_1 + v_1(x) = -v_1(L) + v_1(x)
\]  

(2.37)

\[
= -\frac{1}{L} \int_0^L f(x_0)(L - x_0) \, dx_0 + \frac{1}{L} \int_0^x f(x_0)(L - x_0) \, dx_0
\]  

(2.38)

\[
= -\frac{1}{L} \int_x^L f(x_0)(L - x_0) \, dx_0.
\]  

(2.39)

\(^4\)In reality all problems have to have some time dependence of course. What usually happens is that after some initial transients have decayed we are left with a steady state solution which may or may not be the trivial one \( u = 0 \).
We can therefore write

$$u(x) = \frac{x}{L} \int_x^L (x_0 - L)f(x_0) \, dx_0 + \frac{(x - L)}{L} \int_x^x x_0 f(x_0) \, dx_0$$

Finally this means we can write the solution in the form

$$u(x) = \int_0^L G(x,x_0)f(x_0) \, dx_0 \quad (2.40)$$

where

$$G(x,x_0) = \begin{cases} x_0(x - L), & 0 \leq x_0 \leq x, \\ \frac{x}{L}(x_0 - L), & x \leq x_0 \leq L. \end{cases} \quad (2.41)$$

The function $G(x,x_0)$ can be thought of as an “influence function”. It is in fact the Green’s function for this problem and we will say more about this later on. Note that $G(x,x_0) = \overline{G(x_0,x)} = G(x_0,x)$ here, i.e. it is symmetric (the overline or “bar” denotes the complex conjugate, recall $z = a + ib, \overline{z} = a - ib$). The Green’s function does not always possess this full symmetry; it only occurs for special types of boundary value problems. In particular $G(x,x_0) = \overline{G(x_0,x)}$ always occurs for a special class of problems called self-adjoint operator problems (which we will consider shortly).

Note that we may write (2.41) in the form

$$G(x,x_0) = \frac{x}{L}(x_0 - L)H(x_0 - x) + \frac{x_0}{L}(x - L)H(x - x_0) \quad (2.42)$$

which also illustrates the symmetry, where

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$$

is the so-called Heaviside step function.

When determining Green’s function later, I would always encourage you to write them in this form. It helps a great deal, especially when integrating them!

Finally we note that by directly integrating twice we could in fact obtain the solution in the form (see question 5 on Example Sheet 1)

$$u(x) = \int_0^x \int_0^{x_0} f(x_1) \, dx_1 dx_0 + c_1 x + c_2. \quad (2.43)$$

You are asked to show that this is equivalent to (2.40) in question 5 on Example Sheet 1.
2.3 Linear differential operators

It turns out to be very useful to define the notation $\mathcal{L}$ to mean a linear operator, which means that

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L} u_1 + c_2 \mathcal{L} u_2.$$ 

for (possibly complex) constants $c_j$. In this chapter it will be associated with a second order ordinary differential operator, e.g. $\mathcal{L} = d^2/dx^2$. In the next chapter it will be associated with partial differentiation. Remember that in general an operator will take a function and turn it into another function. The functions in general will belong to some function space which possess some specific properties, i.e. $L^2[a,b]$ which means that they are square integrable on $[a,b]$, (i.e. $f \in L^2[a,b]$ means $\int_a^b |f(x)|^2 \, dx < \infty$) etc.

We are interested in the linear BVP

$$\mathcal{L} u = p(x) \frac{d^2 u}{dx^2} + r(x) \frac{du}{dx} + q(x) u = f(x)$$  \hspace{1cm} (2.44)

where for now we do not make any restrictions on the functions $p(x), r(x), q(x)$ and $f(x)$ but they can be complex functions and we usually consider them as continuous. The (real) domain on which the ODE holds is $x \in [a,b]$ and it is of course subject to BCs on $x = a, b$ which we shall denote as $\mathcal{B}$. We will restrict attention to homogeneous BCs and for now these could be of any form, e.g.

$$\mathcal{B} = \{ u(a) = 0, u(b) = 0 \}, \quad \text{Dirichlet} \hspace{1cm} (2.45)$$
$$\mathcal{B} = \{ u'(a) = 0, u(b) = 0 \}, \quad \text{Dirichlet-Neumann}, \hspace{1cm} (2.46)$$
$$\mathcal{B} = \{ u'(a) = 0, u'(b) = 0 \}, \quad \text{Neumann}, \hspace{1cm} (2.47)$$
$$\mathcal{B} = \{ u(a) + hu'(a) = 0, u(b) = 0 \}, \quad \text{Robin-Dirichlet}, \hspace{1cm} (2.48)$$
$$\mathcal{B} = \{ u(a) = u(b), u'(a) = u'(b) \}, \quad \text{Periodic}, \hspace{1cm} (2.49)$$
$$\mathcal{B} = \{ u(a) + hu'(b) = 0, u(b) = 0 \}, \quad \text{Mixed-Dirichlet}. \hspace{1cm} (2.50)$$

Extension to the case of inhomogeneous BCs is not too difficult - we shall discuss this in section 2.12.

The BVP therefore consists of the equation $\mathcal{L} u = f(x)$ and the BCs $\mathcal{B}$.

2.3.1 Inner products

The function spaces to which the functions that we are interested in belong, are endowed with an inner product. This means that they are “inner product spaces”. This basically means that they possess nice properties such as Cauchy-Schwarz and the triangle inequality. We do not worry too much about this here, usually assuming that the functions we are interested in are in $L^2[a,b]$. The notion and notation of an inner product is useful. We define the usual inner product as

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) \, dx$$ \hspace{1cm} (2.51)

where we note that $\overline{f(x)}$ denotes the complex conjugate of the function $f$, i.e. we have defined this inner product over the set of complex valued functions (this includes the set of real functions of course).
We have the important properties of inner product spaces that
\[ \langle f, g \rangle = \langle g, f \rangle, \]  
\[ \langle f, \alpha g_1 + \beta g_2 \rangle = \alpha \langle f, g_1 \rangle + \beta \langle f, g_2 \rangle, \]  
\[ \langle f, f \rangle \geq 0 \text{ with equality if and only if } f = 0. \]  
\[ \langle \alpha g_1 + \beta g_2, f \rangle = \overline{\alpha} \langle g_1, f \rangle + \overline{\beta} \langle g_2, f \rangle. \]  
(2.52)  
(2.53)  
(2.54)  
(2.55)

2.3.2 The adjoint operator

It is useful to define a so-called adjoint BVP associated with the original BVP above. This adjoint problem consists of an adjoint operator \( \mathcal{L}^* \) and associated adjoint BCs \( \mathcal{B}^* \). These are defined by
\[ \langle v, \mathcal{L}w \rangle = \langle \mathcal{L}^*v, w \rangle \]
noting that this prescribes both an operator and BCs and in general \( \mathcal{L}^* \neq \mathcal{L} \) and \( \mathcal{B}^* \neq \mathcal{B} \).

Example 2.5 Assuming that \( u, v \in L^2[a, b] \) (i.e. they are square integrable on \([a, b]\)), find the adjoint operator and BCs for the following problems

\[ (i) \quad \mathcal{L} = \frac{d^2}{dx^2}, \quad \mathcal{B} = \{ u(0) = 0, u(1) = 0 \}, \]  
\[ (ii) \quad \mathcal{L} = \frac{d^2}{dx^2} + \frac{d}{dx} + 1, \quad \mathcal{B} = \{ u(0) = 0, u(1) = 0 \}, \]  
\[ (iii) \quad \mathcal{L} = \frac{d^2}{dx^2} + 1, \quad \mathcal{B} = \{ u(0) = 0, u(1) = u'(0) \}, \]  
\[ (iv) \quad \mathcal{L} = \frac{d^2}{dx^2} + 1, \quad \mathcal{B} = \{ u(0) = u(1), u'(0) = u'(1) \}, \]  
\[ (v) \quad \mathcal{L} = \frac{d^2}{dx^2} + i \frac{d}{dx} + 1, \quad \mathcal{B} = \{ u(0) = 0, u(1) = 0 \}, \]  
\[ (vi) \quad \mathcal{L} = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + 1, \quad \mathcal{B} = \{ u(1) = 0, u'(2) = 0 \}, \]  
\[ (vii) \quad \mathcal{L} = \frac{d^2}{dx^2} + k^2, \quad \mathcal{B} = \{ u'(x) \pm iku(x) \to 0 \text{ as } x \to \pm \infty \}. \]  
(2.56)  
(2.57)  
(2.58)  
(2.59)  
(2.60)  
(2.61)  
(2.62)

The trick is to use integration by parts to interchange the order of integration onto the “other” function.

(i) Let us follow through the argument, using integration by parts:
\[ \langle v, \mathcal{L}u \rangle = \int_0^1 \frac{d^2u}{dx^2} \frac{dv}{dx} \, dx \]
\[ = \left[ \frac{dv}{dx} \right]_0^1 - \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx \]
\[ = \left[ \frac{dv}{dx} \right]_0^1 - \left[ \left[ \frac{du}{dx} \right]_0^1 - \int_0^1 \frac{d^2v}{dx^2} \, dx \right] \]
\[ = \left[ \frac{dv}{dx} \right]_0^1 + \int_0^1 \frac{d^2v}{dx^2} \, dx \]
\[ = \left[ \frac{dv}{dx} \right]_0^1 + \langle \mathcal{L}^*u, v \rangle \]  
(2.63)
where $\mathcal{L}^* = d^2/dx^2$ and in the last step we have imposed the BCs on $u$. In order to ensure that the term in brackets is zero we must choose $B^* = \{\tau(0) = \tau(1) = 0\}$ but this is equivalent to having $v(0) = v(1) = 0$ (If $v$ is a complex function then it being zero means both its real and imaginary parts must be zero and hence these conditions are equivalent). We see that $\mathcal{L}^* = \mathcal{L}$ and the adjoint BCs are the same as the original BCs, i.e. $B^* = B$.

(ii) The first term of the operator is identical with that in (i) so we can use that result.

$$
\langle v, \mathcal{L}u \rangle = \int_0^1 v \left( \frac{d^2u}{dx^2} + \frac{du}{dx} + u \right) \, dx
$$

$$
= \left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 + \int_0^1 \frac{d^2v}{dx^2} u \, dx + \int_0^1 \frac{dv}{dx} \, dx + \int_0^1 v \, dx
$$

$$
= \left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 + \int_0^1 \left( \frac{d^2v}{dx^2} - \frac{dv}{dx} + v \right) u \, dx
$$

$$
= \left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 + \langle \mathcal{L}^* v, u \rangle
$$

and we note here that $\mathcal{L}^* \neq \mathcal{L}$ due to the first derivative term. The adjoint BCs are unchanged however, $B^* = B$.

(iii) Using (i) above it is easily shown that

$$
\langle v, \mathcal{L}u \rangle = \int_0^1 \frac{d^2u}{dx^2} + u \, dx
$$

$$
= \left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 + \int_0^1 \left( \frac{d^2v}{dx^2} + v \right) \, dx
$$

$$
= \left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 + \langle \mathcal{L}^* v, u \rangle.
$$

so that $\mathcal{L}^* = \mathcal{L}$. Let us now determine the adjoint BCs, $B^*$. We need

$$
\left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 = 0
$$

and using the BCs $u(0) = 0$ and $u(1) = u'(0)$ we see that

$$
\left[ \frac{d}{dx} \left( \frac{d}{dx} u \right) - u \frac{d}{dx} \right]_0^1 = (v(1)u'(1) - u(1)v'(1)) - (v(0)u'(0) - u(0)v'(0)),
$$

$$
= v(1)u'(1) - u(1)v'(1) - v(0)u'(0),
$$

$$
= v(1)u'(1) - (v(1) + v(0))u'(0)
$$

(2.67)

which implies that we require the adjoint BCs to be

$$
v(1) = 0, \quad v'(1) = -v(0).
$$

(2.68)

Note in particular that in this example, although $\mathcal{L}^* = \mathcal{L}$ the adjoint BCs are different from the original BCs, $B^* \neq B$.

(iv)-(vii) See question 5 on Example Sheet 2.
In question 6 of Example Sheet 2 you are asked to show that the adjoint operator associated with the general ODE \((2.44)\) is

\[
L^* = p(x) \frac{d^2}{dx^2} + \left(2\frac{dp}{dx} - r\right) \frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right),
\]  
(2.69)

**Lagrange’s\(^5\) identity**

Lagrange derived a very useful identity. This is:

\[
vl \cdot L u - L^* v u = \frac{d}{dx} \left[ p \left( \frac{du}{dx} - u \frac{dv}{dx} \right) + \left( r - \frac{dp}{dx} \right) u v \right]
\]  
(2.70)

You are asked to prove this in question 7 on Example Sheet 2.

**Green’s\(^6\) second identity**

We can integrate both sides of Lagrange’s identity \((2.70)\) between \(x = a\) and \(x = b\) to get

\[
\int_a^b vl \cdot L u - L^* v u \; dx = \left[ p \left( \frac{du}{dx} - u \frac{dv}{dx} \right) + \left( r - \frac{dp}{dx} \right) u v \right]_a^b.
\]  
(2.71)

Note that this general identity is very useful in order to determine the adjoint BCs \(B^*\) required above.

With inner product notation we note that we can write \((2.71)\) as

\[
\langle v, Lu \rangle - \langle L^* v, u \rangle = \left[ p \left( \frac{du}{dx} - u \frac{dv}{dx} \right) + \left( r - \frac{dp}{dx} \right) u v \right]_a^b.
\]  
(2.72)

I had a sentence here which referred to “real function spaces”; please delete and ignore - it was very confusing and did not add anything! Apologies.

### 2.3.3 Self-adjoint operators

Self-adjoint (S-A) operators are special operators with the property that the adjoint problem is identical to the original problem, i.e. both the adjoint operator and the adjoint BCs are the same as the original physical BVP. I.e. \(L^* = L\) and \(B^* = B\). E.g. Example 2.5(i) above. It is sometimes the case that the differential operator is the same, i.e. \(L^* = L\) but the boundary conditions are not, e.g. Example 2.5(iii) above. In this case the operator is said to be formally self-adjoint.

---

\(^5\)Joseph-Louis Lagrange (1735-1813) was a brilliant Italian-born French mathematician and astronomer. He made significant contributions in many branches of science, in particular to analysis, number theory, and classical and celestial mechanics. Note that France has an incredible history in mathematics and engineering - if you are ever in Paris, go to the Eiffel Tower and look at the names engraved on each side of the lower part of the tower. You can also see this on wiki: [http://en.wikipedia.org/wiki/List_of_the_72_names_on_the_Eiffel_Tower](http://en.wikipedia.org/wiki/List_of_the_72_names_on_the_Eiffel_Tower) and note that Lagrange is present!

\(^6\)We have already mentioned Green - he was the Nottingham miller!
Example 2.6  Referring to Example 2.5 above, determine which of (i)-(vi) are self adjoint.

(i) \( L^* = L \) and \( B^* = B \). So self-adjoint.
(ii) \( L^* \neq L \) so not self-adjoint.
(iii) Although \( L^* = L \), \( B^* \neq B \) so not self-adjoint. (This is called only formally self-adjoint)
(iv)-(vii) See Question 5 on Example Sheet 2.

In general, mixed BCs do not lead to self-adjoint operators, although if \( p(x) = \text{constant} \), then periodic BCs (which are mixed) do yield self-adjoint operators.

For complex linear operators (i.e. where \( p, r \) and \( q \) are complex functions, the conditions for self-adjointness are complicated. They are in fact that \( p(x) \) has to be a real function with \( p' = \text{Re}(r) \) and \( 2\text{Im}(q) = (\text{Im}(r))' \) (here Re and Im denote the real and imaginary parts of the function respectively. See Question 8 on Example Sheet 2.

For simplicity let us restrict attention from now on to real operators, so that \( p, q \) and \( r \) are real functions. Of course the functions \( u \) and \( v \) could still be complex. We see then from the form of the general adjoint operator in (2.69) that a necessary condition for a second order differential operator to be formally self-adjoint (i.e. \( L^* = L \)) is that \( r(x) = p'(x) \). The operator can then be written as

\[
\mathcal{L}u = \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u(x). \tag{2.73}
\]

In this case Green’s second identity simplifies to

\[
\langle v, \mathcal{L}u \rangle - \langle \mathcal{L}v, u \rangle = \left[ p(x)(v(x)u'(x) - u'(x)v(x)) \right]_a^b \tag{2.74}
\]

and in order to derive the adjoint BCs \( B^* \) satisfied by \( v \) we choose them such that the right hand side of (2.74) is zero. For a given \( B \) satisfied by \( u \) this defines the conditions \( B^* \) satisfied by \( v \). This also shows that even if \( L^* = L \), we may not have \( B^* = B \). We therefore reiterate here that the property of self-adjointness requires properties of BCs, not just the operator itself. In particular it could be that the operator is formally self-adjoint so that \( L^* = L \) but the required adjoint BCs in order to ensure that (2.74) is satisfied are not the same as the original BCs.

2.3.4 Forcing formal self-adjointness

In fact we can use what we know about first order ODEs in order to write all second order ODEs in a formal self-adjoint form as we show in section 2.11. However, even though we can do this, we note that the BCs may not lead to a fully self-adjoint operator.

Let us now consider a very special type of BVP, the so-called Sturm-Liouville problems.
2.4 Sturm-Liouville (S-L) eigenvalue problems

The problems that we will be concerned with in this section are the so-called Sturm-Liouville\(^7\) ODE BVPs which take the form of an operator in S-A form, i.e.

\[
\mathcal{L}u = \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u(x) \quad (2.75)
\]

with \(x \in [a, b]\): this could also be the whole real line or the semi-infinite domain, e.g. \(x \in [0, \infty)\). The functions \(p, q\) and \(\mu\) are real and continuous. In general \(p\) is non-negative (and usually positive almost everywhere) and \(\mu\) is positive. We will associate some homogeneous BCs with this ODE shortly.

If the operator if NOT in the form (2.75), the problem is NOT a S-L problem.

Naturally arising problems in the physical sciences often lead to the equation

\[
\mathcal{L}\phi(x) + \lambda \mu(x) \phi(x) = 0 \quad (2.76)
\]

where \(\mu(x)\) arises via the physics in the derivation of the governing equations. This is accompanied by boundary conditions. Solutions to this problem exist only for particular values of \(\lambda\) say \(\lambda_k\) (the eigenvalues), for \(k = 1, 2, 3, \ldots\), with associated solution \(\phi_k(x)\) (the eigenfunctions). The eigenvalues and eigenfunctions are usually of great physical interest and significance.

Regular S-L problem

The regular Sturm-Liouville eigenvalue problem is defined by the ODE

\[
\mathcal{L}\phi(x) + \lambda \mu(x) \phi(x) = 0 \quad (2.77)
\]

with \(\mathcal{L}\) as defined in (2.75) and homogeneous boundary conditions of the form

\[
\mathcal{B} = \{ \alpha_1 \phi(a) + \alpha_2 \frac{d\phi}{dx}(a) = 0, \beta_1 \phi(b) + \beta_2 \frac{d\phi}{dx}(b) = 0 \}, \quad (2.78)
\]

where \(\alpha_n, \beta_n\) are real, \(x \in [a, b]\) (a finite interval), the functions \(p(x), q(x)\) and \(\mu(x)\) are real and continuous, \(p'(x)\) exists and is continuous, and \(p(x), \mu(x)\) are positive.

We note that the BCs here are not mixed. This is important as we shall see later.

Also, note that the fact that \(\alpha_n\) and \(\beta_n\) are real ensures the self-adjointness (i.e. full S-A not just formal) of the problem: Regular S-L problems are fully self-adjoint! (but note the many conditions required for regularity!)

Singular S-L problem

We sometimes want to relax the conditions above since physical problems are often not quite as constrained. We will not be too prescriptive here about the type of non-regular

\(^7\)named after the French mathematicians Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1893) who studied these in the early 19th century. This work was very influential for the theory of ODEs.
S-L problem we consider but will occasionally refer to them as we proceed. What often happens in singular S-L problems is that e.g. \( p(x) \) vanishes at one of the end points of the interval \([a, b]\) or e.g. the boundary conditions are not quite of the form in (2.78), e.g. periodic conditions with \( p = \text{constant} \).

## 2.4.1 Theorems associated with Regular S-L problems for ODEs

For a regular S-L ODE problem we have the following important theorems:

1. All eigenvalues \( \lambda \) are real

2. There are an infinite number of eigenvalues

\[
\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots
\]  

(2.79)

There is a smallest eigenvalue \( \lambda_1 \) but no largest eigenvalue: \( \lambda_n \to \infty \) as \( n \to \infty \).

3. Corresponding to each eigenvalue \( \lambda_n \) there is an eigenfunction say \( \phi_n(x) \) which is unique to within an arbitrary multiplicative constant. \( \phi_n(x) \) has \( n - 1 \) zeros for \( x \in (a, b) \).

4. The eigenfunctions form a complete set. This means that any piecewise smooth function \( g(x) \) can be represented in the form

\[
g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)
\]  

(2.80)

Importantly this series is convergent, converging to \( (g(x^+) + g(x^-))/2 \) where \( x^+ \) and \( x^- \) denote approaching \( x \) from above and below respectively. Thus for continuous functions this series converges to \( g(x) \).

5. Eigenfunctions associated with different eigenvalues are orthogonal relative to the weight function \( \mu(x) \). I.e. if \( \lambda_m \neq \lambda_n \ (m \neq n) \)

\[
\int_a^b \mu(x) \phi_m(x) \phi_n(x) = 0
\]  

(2.81)

If the S-L problem is singular, these theorems may still hold, but not necessarily.

Since this is a course on Green’s functions rather than ODEs, we do not go into the details of these theorems too much. Although let us discuss a simple example to illustrate their usefulness in a simple important case.
2.4.2 A model example to illustrate the theorems

Example 2.7 We set $p = 1, q = 0$ in (2.75) and thus consider the associated eigenvalue problem for the Laplacian operator in one dimension, with the weighting $\mu(x) = 1$. These eigenfunctions are therefore appropriate for the heat equation and wave problems in one space dimension as you will have seen in MT20401. The eigenfunction equation is

$$\phi''(x) + \lambda \phi(x) = 0$$

(2.82)

for $x \in [0, L]$. Let us consider the case when $\mathcal{B} = \{ \phi(0) = 0, \phi(L) = 0 \}$. This is therefore a regular S-L problem.

The solutions of this problem take the form (see question 2 on Example Sheet 3)

$$\phi_n(x) = \sin \left( \frac{n\pi x}{L} \right), \quad \lambda_n = \left( \frac{n\pi}{L} \right)^2$$

(2.83)

with $n = 1, 2, ..., \infty$ and therefore the solution is of the form of a Fourier sine series:

$$u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

(2.84)

for some real coefficients $a_n$ (i.e. $u(x)$ is a real function).

Real eigenvalues

In determining this result you usually assume real eigenvalues. Seeking complex ones can be hard! This theorem tells us that once we have found all of the real eigenvalues we can stop as there are no complex ones!

Eigenvalue ordering

We see that indeed we have an infinite number of eigenvalues $\lambda_n = (n\pi/L)^2$ and that indeed we have a smallest: $(\pi/L)^2$, but no largest.

Zeros of eigenfunctions

Eigenfunctions $\phi_n(x) = \sin \left( \frac{n\pi x}{L} \right)$ should have $n - 1$ zeros inside $(a, b)$. This is clearly true.

Eigenfunction convergence

The eigenfunction expansion (2.84) is a Fourier Sine series and we know (from MT20401) via Fourier’s convergence theorem that any piecewise smooth function can be represented as so. Remember that this helped in MT20401 as we could use separation of variables successfully in many cases.
Eigenfunction orthogonality

The weight function \( \mu(x) \) here is simply unity. We can use the inner product notation and we know that if \( m \neq n \)

\[
\langle \phi_n, \phi_m \rangle = \int_0^L \sin(n \pi x/L) \sin(m \pi x/L) = 0.
\]

Orthogonality of the eigenfunctions enables the coefficients \( a_n \) to be determined in a straightforward manner as

\[
a_n = \frac{\langle u(x), \phi_n \rangle}{\langle \phi_m, \phi_m \rangle} = \frac{\int_0^L u(x) \phi_n \, dx}{\int_0^L \phi_m^2(x) \, dx}.
\]

2.4.3 Proofs of S-L Theorems 1. and 5.

Some of the Theorems 1-5 above relating to S-L problems are difficult to prove. Two of them are relatively simple however: Theorem 1 pertaining to real eigenvalues and Theorem 5 pertaining to orthogonal eigenfunctions. For reasons that will become clear shortly, we will prove Theorem 5 first.

**Theorem 5 - A modified inner product and orthogonal eigenfunctions**

Take two eigenfunctions, \( \phi \) and \( \psi \) (associated with a regular S-L operator \( L \)) corresponding to distinct eigenvalues \( \lambda \) and \( \nu \) say,

\[
L\phi = -\lambda \mu(x)\phi(x), \quad L\psi = -\nu \mu(x)\psi(x).
\]  

(2.85)

Since the operator is regular S-L, it is S-A so that

\[
0 = \langle L\phi, \psi \rangle - \langle \phi, L\psi \rangle,
\]

(2.86)

\[
= -\lambda \langle \mu \phi, \psi \rangle + \nu \langle \phi, \mu \psi \rangle,
\]

(2.87)

\[
= -\lambda \langle \mu \phi, \psi \rangle + \nu \langle \phi, \mu \psi \rangle,
\]

(2.88)

\[
= (\nu - \lambda) \int_a^b \mu(x) \overline{\phi(x)} \psi(x) \, dx
\]

(2.89)

and therefore since the eigenvalues are distinct, using standard inner product notation

\[
\int_a^b \mu(x) \overline{\phi(x)} \psi(x) \, dx = \langle \phi, \mu \psi \rangle = 0.
\]

I.e. the weighted eigenfunctions are orthogonal with respect to the usual inner product defined in (2.51).

Given the above however, it is convenient to define a modified inner product

\[
\langle f, g \rangle = \int_a^b \mu(x) \overline{f(x)} g(x) \, dx.
\]

(2.90)

Then the eigenfunctions themselves are orthogonal with respect to this newly defined inner product. **Unless otherwise stated, we assume that the weighting \( \mu(x) = 1 \).**
Theorem 1 - Real eigenvalues

Take the eigenvalue $\lambda$ corresponding to the eigenfunction $\phi(x)$ associated with a regular S-L operator $L$. We have, working with the modified inner product (2.90) above,

$$\langle L\phi, \phi \rangle = -\langle \lambda \phi, \phi \rangle, \quad (2.91)$$

$$= -\lambda \langle \phi, \phi \rangle. \quad (2.92)$$

Also we have

$$\langle \phi, L\phi \rangle = \langle \phi, -\lambda \phi \rangle, \quad (2.93)$$

$$= -\lambda \langle \phi, \phi \rangle. \quad (2.94)$$

Therefore, since problem is S-A,

$$0 = \langle L\phi, \phi \rangle - \langle \phi, L\phi \rangle, \quad (2.95)$$

$$= (\lambda - \overline{\lambda}) \langle \phi, \phi \rangle \quad (2.96)$$

so that

$$\lambda = \overline{\lambda}$$

and therefore the eigenvalues must be real.

It transpires that this result holds for regular S-L problems, singular S-L problems in the sense that $p(x) = 0$ at an end point, and also if the BCs are periodic.
2.5 Existence and uniqueness of BVPs for ODEs: The Fredholm Alternative

Recall the following theorem for Initial Value Problems associated with ODEs:

**Theorem 2.1** Given the ODE
\[ u''(t) + p(t)u'(t) + q(t)u(t) = f(t) \]
subject to ICs \( u(t_0) = x_0, \ u'(t_0) = v_0 \), if \( p(t), q(t) \) and \( f(t) \) are continuous on the interval \([a,b]\) containing \( t_0 \), the solution of the IVP exists and is unique.

Unfortunately the situation is not as simple for BVPs. It can be the case that BVPs have (i) no solution, (ii) a unique solution or (iii) infinitely many solutions! Let us first state the following theorem which guarantees the existence of two fundamental solutions to a homogeneous ODE:

**Theorem 2.2** Given the homogeneous ODE
\[ p(x)u''(x) + r(x)u'(x) + q(x)u(x) = 0, \]
with \( p, r \) and \( q \) continuous and \( p \) never zero on the domain of interest, there always exist two fundamental solutions \( u_1(x) \) and \( u_2(x) \) which generate the general solution \( u(x) = c_1u_1(x) + c_2u_2(x) \).

Therefore whether a solution exists or not depends on the BCs. As a very simple example to illustrate that BVPs can have a unique solution, no solution or infinitely many solutions, let us consider the following problem.

**Example 2.8** Consider the homogeneous ODE
\[ u''(x) + u(x) = 0 \]
subject to inhomogeneous BCs

(i) \( u(0) = 1, \ u(\pi) = 1 \),
(ii) \( u(0) = 1, \ u(\pi/2) = 1 \),
(iii) \( u(0) = 1, \ u(2\pi) = 1 \).

The fundamental solutions are \( \cos x \) and \( \sin x \) so that \( u(x) = c_1 \cos x + c_2 \sin x \).

The BCs in (i) are inconsistent and therefore there is no solution.

The BCs in (ii) yield the unique solution \( u(x) = \cos x + \sin x \).

The BCs in (iii) yield the infinite family of solutions \( u(x) = \cos x + c_2 \sin x \) where \( c_2 \) is arbitrary.
Let us now consider the case of an inhomogeneous ODE subject to homogeneous BCs (recall that this is the main thrust of our enquiries in this course). We are able to state a rather general theorem regarding existence and uniqueness of solutions to this problem. We consider the additional effect of inhomogeneous BCs in section 2.12. We shall consider an example which illustrates the main issues that arise.

**Example 2.9** Find the solution to the ODE

\[ u''(x) + u(x) = f(x) \]

subject to \( u(0) = u(L) = 0 \).

Fundamental solutions of the homogeneous ODE are \( \sin x \) and \( \cos x \) but remember that the general solution can be any linear combination of these and it is convenient to use \( u_1(x) = \sin x \) and \( u_2(x) = \sin(x - L) \) (since \( \sin(x - L) = \sin x \cos L - \cos L \sin x \)). This is convenient since they satisfy the left and right BCs respectively.

Let us therefore write the solution to the homogeneous problem as \( u(x) = c_1 \sin x + c_2 \sin(x - L) \). We then know from (2.29) that the solution to the inhomogeneous problem can be written

\[ u(x) = (c_1 + v_1(x)) \sin x + (c_2 + v_2(x)) \sin(x - L) \tag{2.97} \]

where

\[ v_1(x) = \int_a^x -\frac{u_2(x_0)f(x_0)}{p(x_0)W(x_0)} \, dx_0 = -\int_0^x \frac{\sin(x_0 - L)f(x_0)}{\sin L} \, dx_0, \tag{2.98} \]

\[ v_2(x) = \int_a^x \frac{u_1(x_0)f(x_0)}{p(x_0)W(x_0)} \, dx_0 = \int_0^x \frac{\sin x_0f(x_0)}{\sin L} \, dx_0. \tag{2.99} \]

noting that \( W(x) = \sin x \cos(x - L) - \sin(x - L) \cos x = \sin(x - (x - L)) = \sin L \) is a constant.

Now impose boundary conditions, with \( u(0) = 0 \) giving

\[ c_2 = 0, \tag{2.100} \]

whilst \( u(L) = 0 \) gives

\[ c_1 \sin L = \int_0^L \sin(x_0 - L)f(x_0) \, dx_0. \tag{2.101} \]

This last equation giving \( c_1 \) is perfectly valid, unless \( L = n\pi \) which knocks out the left hand side! In that case there is then only a solution if

\[ \int_0^{n\pi} f(x_0) \sin(x_0 - n\pi) \, dx_0 = (-1)^n \int_0^{n\pi} f(x_0) \sin x_0 \, dx_0 = 0. \]

Take, e.g. \( L = \pi \). Even if this condition is satisfied then there are infinitely many solutions because we can add on any multiple of \( \sin x \) to the solution, i.e.

\[ u(x) = u_{PS}(x) + c \sin x \]

Often the only solution to the homogeneous BVP is the zero solution. When \( L = \pi \) above we see that \( \sin x \) is a non trivial solution to the homogeneous BVP. This corresponds to an existence of a so-called zero eigenvalue. Interestingly, this tells us something very special about the existence and uniqueness of the solution to the inhomogeneous problem as we now describe via a general theorem. We will return to the example above after we have stated the theorem to see how it aligns with the theorem.
The Fredholm Alternative for ODE BVPs

We can state the following theorem

**Theorem 2.3** We introduce the BVP consisting of the linear ODE

\[ Lu = p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x) \]

subject to homogeneous BCs \( \mathcal{B} \) with \( p(x), r(x), q(x) \) and \( f(x) \) real and continuous on the interval \([a, b]\), with \( p(x) \neq 0 \) on \([a, b]\). Consider also the associated homogeneous adjoint problem

\[ L^*v = 0 \]

with associated homogeneous BCs \( \mathcal{B}^* \).

Then **EITHER**

1. If the only solution to the homogeneous adjoint problem is the trivial solution \( v(x) = 0 \) then the solution to the inhomogeneous problem \( u(x) \) exists and is unique

   OR

2. If there are non-trivial solutions to the homogeneous adjoint problem \( v(x) \neq 0 \) then either
   - There are infinitely many solutions if \( \int_a^b v(x)f(x) = 0 \),
   - or
   - There is no solution if \( \int_a^b v(x)f(x) \neq 0 \).

See question 1 of Example Sheet 4 for some more details of this theorem.

Clearly if the problem is self-adjoint then \( L^* = L \) and \( \mathcal{B}^* = \mathcal{B} \) and so the adjoint homogeneous problem is simply the homogeneous version of the original BVP.

**Example 2.10** How is the Fredholm Alternative Theorem consistent with example 2.9?

Firstly the BVP is S-A and so the adjoint problem is merely the homogeneous version of the original problem. It therefore has solution

\[ v(x) = d_1 \sin x + d_2 \cos x. \]

Imposing \( v(0) = 0 \) yields \( d_2 = 0 \) and \( v(L) = 0 \) gives

\[ d_1 \sin L = 0 \quad (2.102) \]

which means that if \( L \neq n\pi \) we need \( d_1 = 0 \) and therefore the only solution to this problem is the trivial one \( v(x) = 0 \). From the Fredholm Alternative Theorem, the solution \( u(x) \) to the original problem is unique.

If \( L = n\pi \) then (2.102) is trivially satisfied for any \( d_1 \). Therefore a non-trivial solution to the homogeneous adjoint problem is

\[ v(x) = \sin x \]
which from the Fredholm Theorem means that if
\[ \int_0^L \sin x_0 f(x_0) \, dx_0 = 0 \]
there are infinitely many solutions to the original problem, whereas if
\[ \int_0^L \sin x_0 f(x_0) \, dx_0 \neq 0 \]
there are no solutions. This corresponds exactly to the Example above.

The existence of a non-trivial solution to the homogeneous problem corresponds to the existence of a zero eigenvalue. We will see later in section 2.13 that when this happens, the standard Green’s function (as we will define shortly) does not exist and a modified form has to be considered.

One final point. This theorem allows us to say a great deal about the existence and uniqueness of solutions to inhomogeneous ODEs without actually having to solve the problems! We illustrate this with an example.

**Example 2.11** For the following ODE/BC pairings use the Fredholm Alternative to state if a solution exists and if so if it is unique (note that you do not solve the inhomogeneous BVP in order to show this!).

\[ u''(x) + \psi u(x) = \sin x \]

with

(a) \( \psi = 1 \), \( \mathcal{B} = \{u(0) = 0, u(\pi) = 0\} \)
(b) \( \psi = 1 \), \( \mathcal{B} = \{u'(0) = 0, u'\pi = 0\} \)
(c) \( \psi = -1 \), \( \mathcal{B} = \{u(0) = 0, u(\pi) = 0\} \)
(d) \( \psi = 2 \), \( \mathcal{B} = \{u(0) = 0, u(\pi) = 0\} \)

All problems are self-adjoint.

(a) A non-trivial solution to the homogeneous problem is \( v(x) = \sin x \). But we note that

\[ \int_0^\pi \sin^2 x \, dx \neq 0 \]

so therefore a solution does not exist. (Verify this yourself by trying to solve the inhomogeneous problem).

Parts (b)-(d) are considered in question 2 on Example Sheet 4.
The Fredholm Alternative for Linear Systems

As perhaps should be expected, the Fredholm Alternative is far more general than just governing ODEs.

**Theorem 2.4** We introduce the linear system

\[ Lu = f \]

where \( L \) is an \( m \times n \) matrix and \( u \) and \( f \) are \( 1 \times n \) vectors where \( f \) is given and \( u \) is unknown. Consider the homogeneous adjoint (transpose) problem

\[ L^T v = 0 \]

where superscript \( T \) denotes the transpose of the matrix. Then \textbf{EITHER}

1. If the only solution to the homogeneous adjoint problem is the trivial solution \( u = 0 \) then the solution to the inhomogeneous problem \( u \) exists and is unique

   **OR**

2. If there are non-trivial solutions to the homogeneous adjoint problem \( v \neq 0 \) then either

   - There are infinitely many solutions if \( v \cdot f = 0 \),
   - or
   - There is no solution if \( v \cdot f \neq 0 \).

It transpires that this theorem is useful for linear integral equations in later sections.
2.6 What is a Green’s function?

Having addressed many aspects of ODE theory, let us now focus on the main issue of this course - defining and using Green’s functions. The method of Green’s functions is simply a method in order to solve inhomogeneous BVPs. One of the interesting aspects of Green’s functions is that they enable the solution to be written down in a very general form for a variety of forcing functions. The Green’s function also often corresponds to something physically important. We have already seen one example where the Green’s function enables the solution to be written in general form in section 2.2. At that time we did not think of it as a Green’s function, it was considered merely as an “influence” function for the inhomogeneous forcing term \( f(x) \).

2.7 Green’s functions for Regular S-L problems via eigenfunction expansions

Consider again the regular S-L problem of the form

\[
\mathcal{L}u = f(x) \tag{2.103}
\]

with \( \mathcal{L} \) given by (2.75), \( x \in [a, b] \) and \( u \) is subject to two homogeneous BCs of the form (2.78). Also consider the related eigenvalue problem

\[
\mathcal{L}u = -\lambda \mu(x) u \tag{2.104}
\]

with some appropriately chosen \( \mu(x) \). We can solve (2.103) by posing an eigenfunction expansion of the form (see Example Sheet 3)

\[
u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{2.105}\]

This can be differentiated term-by-term (see MT20401) so that, applying \( \mathcal{L} \) we find

\[
\mathcal{L}u(x) = -\sum_{n=1}^{\infty} a_n \lambda_n \mu(x) \phi_n(x) = f(x). \tag{2.106}
\]

Let us multiply by \( \phi_m(x) \) and integrate over the domain \( x \in [a, b] \). The orthogonality of the eigenfunctions (with respect to the weight \( \mu(x) \)) allows us to then show that

\[
-a_n \lambda_n = \frac{\int_a^b f(x) \phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \mu(x) \, dx} \tag{2.107}
\]

Therefore

\[
u(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \left( \frac{-\phi_n(x) \phi_n(x_0)}{\lambda_n \int_a^b \phi_n^2(x_1) \mu(x_1) \, dx_1} \right) \, dx_0 \tag{2.108}\]

and so we recognize that we can write

\[
u(x) = \int_a^b f(x_0) G(x, x_0) \, dx_0 \tag{2.109}\]
where

\[ G(x, x_0) = \sum_{n=1}^{\infty} \left( \frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n \int_a^b \phi_n^2(x_1) \mu(x_1) \, dx_1} \right) \quad (2.110) \]

which is therefore an eigenfunction expansion of the Green’s function. Note that \( G(x, x_0) = G(x_0, x) \) in this setting.

We note that the definition (2.110) would run into difficulty if one of the eigenvalues is zero (i.e. if there is a non-trivial solution to the homogeneous adjoint problem!). We return to this point later on in section 2.13.

**Example 2.12** Let us return to the familiar example

\[ \mathcal{L}u = \frac{d^2u}{dx^2} = f(x) \quad (2.111) \]

with \( u(0) = u(L) = 0 \) and the related eigenvalue problem

\[ \frac{d^2\phi}{dx^2} = -\lambda \phi \quad (2.112) \]

with \( \phi(0) = \phi(L) = 0 \). We already know from example 2.7 that \( \lambda_n = (n\pi/L)^2 \) and \( \phi_n(x) = \sin(n\pi x/L) \) with \( n = 1, 2, 3,... \). Therefore with reference to the theory above, \( u(x) \) is given by

\[ u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \quad (2.113) \]

\[ = \int_0^L f(x_0) G(x, x_0) \, dx_0 \quad (2.114) \]

where

\[ G(x, x_0) = -\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi x/L) \sin(n\pi x_0/L)}{(n\pi/L)^2}. \quad (2.115) \]

Finally we ask, how is this representation of the Green’s function in terms of eigenfunctions related to the form derived in (2.41) or (2.42) above. They must be equivalent! We discuss this in question 4 on Example Sheet 4.
2.8 Green’s functions for Regular S-L problems using a direct approach

For problems of regular (and some singular) S-L type we have shown above in equations (2.109)-(2.110) that the equation

$$\mathcal{L}u = f(x)$$

(2.116)

has the solution

$$u(x) = \int_a^b G(x, x_0) f(x_0) \, dx_0$$

(2.117)

for some appropriately defined function $G(x, x_0)$ which we have termed the Green’s function. We have an eigenfunction representation for the Green’s function defined in (2.110). This approach shows that the Green’s function exists provided that there is no “zero” eigenvalue, see section 2.13. We can obtain the Green’s function for S-L using variation of parameters. We will describe this shortly but first we need some discussion of a few rather unusual “functions”.

2.8.1 The Dirac delta “function”

The representation of the solution in the form (2.117) shows that the source term $f(x)$ represents a forcing at all of the points at which it is non-zero. We can isolate the effect of each point in the following manner. First we take a function $f(x)$ and consider splitting it up in order to take into account the separate contributions from intervals of width $\Delta x_i$ such as we do when carrying out the process of Riemann integration, see figure 4. Consider decomposing the function $f(x)$ into a linear combination of unit pulses starting at the points $x_i$ and being of width $\Delta x_i$, see figure 5.

Figure 4: Figure depicting the partition of a function $f(x)$ into linear contributions of unit pulses, similarly to the process of Riemann integration.
So we would write

\[ f(x) \simeq \sum_i f(x_i) \times \text{(unit pulse starting at } x = x_i). \] (2.118)

and we know that this is only a good approximation if the intervals are small (infinitesimal in fact!)

Indeed this is very similar to something like an integral. Only the \( \Delta x_i \) is missing! Let us now introduce this and a limiting process in the following manner:

\[ f(x) = \lim_{\Delta x_i \to 0} \sum_i f(x_i) \frac{\text{(unit pulse)}}{\Delta x_i} \Delta x_i \] (2.119)

\[ = \lim_{\Delta x_i \to 0} \sum_i f(x_i) (\text{Dirac pulse}) \Delta x_i. \] (2.120)

We now appear to have introduced a strange object - what we have termed here the Dirac pulse. It has height \( 1/\Delta x_i \) and width \( \Delta x_i \). We picture this in figure 6. Note that this pulse has unit area. In the limit as \( \Delta x_i \to 0 \) this pulse represents a concentrated pulse of infinite amplitude located at a single point. It is not really a function but is often termed a generalized function. We will call this object the Dirac Delta function, which when located at the point \( x = x_i \) is written as \( \delta(x - x_i) \). It cannot be written down in the form \( \delta(x - x_i) = \ldots \). We think of this object as a concentrated source or impulsive force, and according to (2.120), in the limit, we have the definition

\[ f(x) = \int_{-\infty}^{\infty} f(x_i) \delta(x - x_i) \ dx_i. \] (2.121)

The interval of integration here is all \( x_i \). The property in (2.121) is known as the sifting property of the Dirac delta function. The dirac delta function can be thought of as the limit of the sequence of various different functions, not only the rectangular type depicted above.

\[ \text{Figure 5: A unit pulse.} \]

We now note some important properties of the function. Firstly, we note that with \( f(x) = 1 \)

\[ 1 = \int_{-\infty}^{\infty} \delta(x - x_i) \ dx_i. \] (2.122)

\[ ^8 \text{Named after the brilliant twentieth century mathematical physicist Paul Dirac (1902-1984)} \]
The function is even, $\delta(x - x_i) = \delta(x_i - x)$. Furthermore it is strongly linked with the Heaviside function $H(x - x_i)$ which we have already defined above, but repeat here for completeness, as

$$H(x - x_i) = \begin{cases} 
1, & x > x_i, \\
0, & x < x_i
\end{cases} \quad (2.123)$$

via the expression

$$H(x - x_i) = \int_{-\infty}^{x} \delta(y - x_i) \ dy. \quad (2.124)$$

The Heaviside function is not defined at $x = x_i$ - we have freedom to choose its value there. Usually the most convenient is to choose its value as $1/2$. This is the average of the limit from both sides of $x_i$ of course.

Finally we note that

$$\delta[c(x - x_i)] = \frac{1}{|c|} \delta(x - x_i) \quad (2.125)$$

for some constant $c$. For proofs of the last two properties see question 6 on Example Sheet 4.

The introduction of this function now allows us to determine an equation governing the Green’s function.

### 2.8.2 Relationship between the Dirac delta function and the Green’s function

Given the solution (2.117), we note that the Green’s function $G(x, x_0)$ is an “influence function” for the source $f(x)$. As an example, let us suppose that $f(x)$ is now a concentrated source at $x = s$, i.e. $f(x) = \delta(x - s)$ with $a < s < b$. This then gives

$$u(x) = \int_{a}^{b} \delta(x_0 - s)G(x, x_0) \ dx_0 = G(x, s) \quad (2.126)$$

by the sifting property (2.121). We therefore obtain the fundamental interpretation of the Green’s function: it is the response at $x$ due to a concentrated source at $x_0$:

$$\mathcal{L}_x G(x, x_0) = \delta(x - x_0) \quad (2.127)$$
where the subscript \( x \) on the operator ensures that we know that the derivatives are with respect to \( x \). The source position \( x_0 \) is a parameter in the problem.

We can check that (2.117) satisfies (2.116) via the definition of the Green’s function (2.127) by operating on each side of (2.117) with \( \mathcal{L}_x \) to give

\[
\mathcal{L}_x u = \int_a^b f(x_0)\mathcal{L}_x [G(x - x_0)] \, dx_0 = \int_a^b f(x_0)\delta(x - x_0) \, dx_0 = f(x)
\]

(2.128)

via the sifting property (2.121).

### 2.8.3 Boundary conditions for the Green’s function BVP

If we take (2.127) together with appropriate homogeneous boundary conditions as an independent definition of the Green’s function (which we shall!) then we also want to derive the solution starting with this independent definition. Start with Green’s identity in 1D for ODEs with operators in S-L form (2.75), which turns out to be

\[
\langle v, \mathcal{L} u \rangle - \langle \mathcal{L} v, u \rangle = \left[ p(x) \left( \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) \right]_a^b.
\]

(2.129)

Let \( v = G(x, x_0) \). The right hand side vanishes as long as we choose the homogeneous BCs for the Green’s function to be the same as those for the original problem associated with \( u \). Then

\[
\langle G(x, x_0), \mathcal{L} u(x) \rangle - \langle \mathcal{L} G(x, x_0), u(x) \rangle = 0.
\]

(2.130)

Using the definition of the Dirac delta function and interchanging variables we obtain

\[
u(x) = \int_a^b f(x_0)G(x_0, x) \, dy.
\]

(2.131)

For regular S-L operators the Green’s function is (Hermitian) symmetric \((G(x_0, x) = G(x, x_0))\) as we shall show shortly, so that

\[
u(x) = \int_a^b f(x_0)G(x, x_0) \, dx_0.
\]

(2.132)

### 2.8.4 Reciprocity and symmetry of the Green’s function for fully S-A problems.

Let us suppose that the BVP is fully self-adjoint (e.g. a regular S-L problem). Let us once again use (2.129) and let \( u = G(x, x_1) \) and \( v = G(x, x_2) \). Both satisfy homogeneous boundary conditions of the form (2.78). Furthermore since \( \mathcal{L}_x u = \delta(x - x_1) \) we use Green’s second identity to find

\[
\int_a^b \left[ G(x, x_2)\delta(x - x_1) - \mathcal{L}_x G(x, x_2)G(x, x_1) \right] \, dx = 0.
\]

(2.133)
Therefore from the sifting property of the Dirac function,

\[
\overline{G}(x_1, x_2) = \int_a^b \overline{L}G(x, x_2)G(x, x_1) \, dx
\]

(2.134)

\[= \int_a^b \overline{L}G(x, x_2)\overline{G}(x_1, x) \, dx \]

(2.135)

\[= \int_a^b \delta(x - x_2)\overline{G}(x_1, x) \, dx \]

(2.136)

\[= \overline{G}(x_2, x_1) \]

(2.137)

\[= G(x_2, x_1) \]

(2.138)

Note that this is all reliant on the fact that the operator is fully self-adjoint. If it is not, much of the above theory has to be modified as we shall see in section 2.10.

Physically, the property (2.138) says that the response at \(x_1\) due to a concentrated source at \(x_2\) is the same as the response at \(x_2\) due to a concentrated source at \(x_1\). This is not immediately physically obvious!

2.8.5 Jump conditions at \(x = x_0\)

The Green’s function can be determined from the governing equation (2.127). For \(x < x_0\), \(G(x, x_0)\) satisfies this equation with a homogeneous BC at \(x = a\). Similarly for \(x > x_0\) with a homogeneous BC at \(x = b\). What happens at the point \(x = x_0\)? We need to consider the type of singularity that arises in (2.127) with reference to the property (2.124).

Suppose firstly that \(G(x, x_0)\) has a jump discontinuity at \(x = x_0\) (a property shared by the Heaviside function \(H(x - x_0)\)). Then \(dG(x, x_0)/dx\) would have a delta function singularity and so \(d^2G(x, x_0)/dx^2\) would be more singular than the actual right hand side of (2.127). Therefore we conclude that \(G(x, x_0)\) must be continuous at \(x = x_0\) which we denote by

\[G(x, x_0)]^x=x_0^+_{x=x_0^-} = 0 \]

(2.139)

where \(x_0^+\) and \(x_0^-\) denote approaching \(x = x_0\) from above and below respectively, e.g. \(x_0^+ = \lim_{\epsilon \to 0^+} x_0 + \epsilon, x_0^- = \lim_{\epsilon \to 0^-} x_0 - \epsilon\) with \(\epsilon > 0\).

On the other hand \(dG(x, x_0)/dx\) does have a jump discontinuity at \(x = x_0\). In order to illustrate this for S-L problems, integrate

\[\mathcal{L}u = f(x)\]

where \(\mathcal{L}\) is the S-L operator (2.75), between \(x = x_0^-\) and \(x = x_0^+\) to give (since \(q\) and \(G\) are continuous at \(x = x_0\))

\[\left[p(x) \frac{dG}{dx}\right]_{x=x_0^+} = 1. \]

(2.140)

Since \(p(x)\) is a continuous function this then gives

\[\left[\frac{dG}{dx}\right]_{x=x_0^-} = \frac{1}{p(x_0)^+} \]

(2.141)
2.8.6 Summary: Green’s function for regular S-L problems

Given a regular S-L problem of the form
\[ \mathcal{L}_x u = \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u(x) = f(x) \] (2.142)
together with homogeneous boundary conditions \( B \) at \( x = a, b \), the corresponding Green’s function will be defined by
\[ \mathcal{L}_x G(x, x_0) = \delta(x - x_0) \] (2.143)
together with the same homogeneous boundary conditions \( B \) at \( x = a, b \), and the following conditions at \( x = x_0 \):
\[ [G(x, x_0)]_{x=x_0}^{x=x_0^+} = 0 \] (2.144)
and
\[ \left[ \frac{dG}{dx} \right]_{x=x_0}^{x=x_0^+} = \frac{1}{p(x_0)} \] (2.145)

Let us use these steps to construct the Green’s function for a simple example.
Example 2.13 Consider again the steady state heat equation

\[ \frac{d^2u}{dx^2} = f(x) \]  \hspace{1cm} (2.146)

with \( u(0) = 0, u(L) = 0 \). We can write the solution to this problem in the form

\[ u(x) = \int_0^L f(x_0)G(x, x_0) \, dx_0 \]  \hspace{1cm} (2.147)

where the Green’s function \( G(x, x_0) \) satisfies

\[ \frac{d^2G(x, x_0)}{dx^2} = \delta(x - x_0) \]  \hspace{1cm} (2.148)

with \( G(0, x_0) = 0 \) and \( G(L, x_0) = 0 \).

Now that we have a governing equation for the Green’s function we can easily obtain its solution for \( x \neq x_0 \):

\[ G(x, x_0) = \begin{cases} \begin{align*} a + bx, & \text{if } x < x_0, \nonumber \end{align*} \\ c + dx, & \text{if } x > x_0 \end{cases} \]  \hspace{1cm} (2.149)

but we note that the “constants” could be different for different \( x_0 \) - the source location.

The BC at \( x = 0 \) applies for \( x < x_0 \) and imposing this gives \( a = 0 \). Similarly \( G(L, x_0) = 0 \) gives \( c + dL = 0 \). Therefore we have

\[ G(x, x_0) = \begin{cases} \begin{align*} bx, & \text{if } 0 \leq x < x_0, \nonumber \end{align*} \\ d(x - L), & \text{if } x_0 \leq x \leq L \end{cases} \]  \hspace{1cm} (2.150)

We also know from the discussion above that \( G(x, x_0) \) is continuous at \( x = x_0 \). This gives

\[ bx_0 = d(x_0 - L). \]  \hspace{1cm} (2.151)

The jump condition on the derivative at \( x = x_0 \), gives (since \( p = 1 \))

\[ d - b = 1. \]  \hspace{1cm} (2.152)

Solve (2.151) and (2.152) to obtain

\[ b = \frac{(x_0 - L)}{L}, \quad d = \frac{x_0}{L} \]  \hspace{1cm} (2.153)

noting in particular the dependence on \( x_0 \) here. This gives

\[ G(x, x_0) = \begin{cases} \begin{align*} \frac{1}{L}(x_0 - L), & \text{if } 0 \leq x \leq x_0, \nonumber \end{align*} \\ \frac{1}{L}(x - L), & \text{if } x_0 \leq x \leq L \end{cases} \]  \hspace{1cm} (2.154)

which agrees with what we found in (2.41).

In fact, for regular (and some singular) S-L problems we can derive the Green’s function explicitly via the method of variation of parameters as we describe in these steps:
2.8.7 Explicit solution for the Green’s function for regular S-L problems

1. Find the two independent solutions of the homogeneous equation (2.142) (i.e. the complementary function $u_c$) say $u_1(x)$ and $u_2(x)$.

2. Take linear combinations of these solutions in order to find a solution which satisfies the left (at $x = a$) and right (at $x = b$) homogeneous boundary conditions. Call these $u_L(x)$ and $u_R(x)$ respectively.

3. Write the Green’s function as

$$G(x, x_0) = \begin{cases} c_L(x_0)u_L(x), & a \leq x \leq x_0, \\ c_R(x_0)u_R(x), & x_0 \leq x \leq b \end{cases}$$

(2.155)

where we have noted the explicit dependence of $c_L$ and $c_R$ on the source location $x_0$. Note that we are able to put $\leq$ here because the Green’s function is continuous at $x = x_0$.

4. Enforce the condition on $G(x, x_0)$ at $x = x_0$:

$$c_L(x_0)u_L(x_0) = c_R(x_0)u_R(x_0)$$

(2.156)

5. Enforce the condition on $dG/dx$ at $x = x_0$:

$$c_R(x_0)\frac{du_R}{dx}(x_0) - c_L(x_0)\frac{du_L}{dx}(x_0) = \frac{1}{p(x_0)}$$

(2.157)

6. Finally we can solve (2.156) and (2.157) for $c_L(x_0)$ and $c_R(x_0)$ to get

$$c_L(x_0) = \frac{u_R(x_0)}{p(x_0)W(x_0)}, \quad c_R(x_0) = \frac{u_L(x_0)}{p(x_0)W(x_0)},$$

(2.158)

where we have defined the associated Wronskian

$$W(x_0) = \begin{vmatrix} u_L(x_0) & u_R(x_0) \\ u'_L(x_0) & u'_R(x_0) \end{vmatrix}.$$  

(2.159)

And therefore the Green’s function is known immediately once $u_L$ and $u_R$ have been determined.

Although such an explicit form is also available when the problem is not of Sturm-Liouville type it needs a little more justification, see section 2.11. Let us first consider some examples which do conform to S-L type.

**Example 2.14** Let us reconsider the steady state heat equation from Example 2.13 where we determined the Green’s function by using a direct method, instead of the variation of parameters procedure above. We note that $p(x) = 1$ and we have the two homogeneous solutions $u_L(x) = x$ and $u_R(x) = x - L$. Therefore since from (2.159) we have $W(x_0) = x_0 - (x_0 - L) = L$, from (2.155) and (2.158) we see that

$$G(x, x_0) = \begin{cases} \frac{1}{L}x(x_0 - L), & 0 \leq x \leq x_0, \\ \frac{1}{L}x_0(x - L), & x_0 \leq x \leq L \end{cases}$$

(2.160)

which agrees with (2.154) which we note in turn agreed with the alternative derivation of (2.41). Note how quickly we could derive the Green’s function with the procedure above!
Example 2.15  Let us consider once again the Green’s function associated with the steady state heat equation

\[ \frac{d^2 u}{dx^2} = f(x) \]  

but now with boundary conditions \( u(0) = 0, u'(L) = 0 \). The Green’s function \( G(x, x_0) \) satisfies

\[ \frac{d^2 G(x, x_0)}{dx^2} = \delta(x - x_0) \]

with \( G(0, x_0) = 0 \) and \( dG/dx(L, x_0) = 0 \).

By the argument above, all we have to do is to find the homogeneous solutions which satisfy separately the left and right boundary conditions and we can immediately write down the Green’s function. These are respectively \( u_L(x) = x \) and \( u_R(x) = 1 \). Therefore \( W(x_0) = -1 \) and

\[
G(x, x_0) = \begin{cases} 
-x, & 0 \leq x \leq x_0, \\
-x_0, & x_0 \leq x \leq L 
\end{cases}
\]

\[ = -xH(x_0 - x) - x_0H(x - x_0) \]  

(2.163)

The next example is an example of a singular S-L problem showing that the method also works in that case.

This is the example that was inconsistent in the lectures. Note that now I have changed the forcing \( f(x) \) so that the problem IS consistent. Note that the actual construction of the Green’s function has not changed.

Example 2.16  Construct the Green’s function associated with the BVP consisting of the ODE

\[ x^2 u''(x) + 2xu'(x) - 2u(x) = f(x) \]  

(2.164)

together with the BCs \( u(0) = 0 \) and \( u(1) = 0 \). Use the Green’s function to find the solution when \( f(x) = x^2 \) and confirm this by finding the solution via standard techniques.

First we note that the ODE is of S-L type since it may be written

\[
\frac{d}{dx} \left( x^2 \frac{du}{dx} \right) - 2u(x) = f(x). 
\]

(2.165)

Therefore we can appeal to all of the theory above.

Since the ODE is of Euler type we seek solutions in the form

\[ u(x) = x^m \]

(2.166)

and therefore we require \( m^2 + m - 2 = 0 \) so that \( (m + 2)(m - 1) = 0 \) and therefore \( m = -2 \) or \( m = 1 \) and therefore we have \( u_1(x) = 1/x^2 \) and \( u_2(x) = x \). These can be combined to yield the solutions

\[ u_L(x) = x, \quad u_R(x) = x - \frac{1}{x^2}. \]  

(2.167)
We find that \( p(x_0)W(x_0) = 3 \) and therefore

\[
\begin{align*}
  c_L(x_0) &= \frac{1}{3} u_R(x_0) = \frac{(x_0^3 - 1)}{3x_0^2}, \\
  c_R(x_0) &= \frac{1}{3} u_L(x_0) = \frac{x_0}{3}
\end{align*}
\]

so that

\[
G(x, x_0) = \begin{cases} 
  \frac{x}{3} \left( \frac{x_0^3 - 1}{x_0^2} \right), & 0 \leq x \leq x_0, \\
  x_0 \left( \frac{x^3 - 1}{x^2} \right), & x_0 \leq x \leq 1,
\end{cases}
\]

\[
= \frac{x}{3} \left( \frac{x_0^3 - 1}{x_0^2} \right) H(x_0 - x) + \frac{x_0}{3} \left( \frac{x^3 - 1}{x^2} \right) H(x - x_0)
\]

The solution is therefore

\[
u(x) = \int_0^1 G(x, x_0) f(x_0) dx_0
\]

and for us since \( f(x_0) = x_0^2 \), using the (Heaviside form of the) Green’s function above,

\[
u(x) = \frac{x}{3} \int_x^1 \left( x_0^3 - 1 \right) dx_0 + \frac{(x^3 - 1)}{3x^2} \int_0^x x_0^3 dx_0
\]

\[
= \frac{x}{3} \left[ \frac{1}{4} x_0^4 - x_0 \right]_x^1 + \frac{1}{12} \left( x - \frac{1}{x^2} \right) x^4
\]

\[
= \frac{x}{3} \left( -\frac{3}{4} - \frac{1}{4} x^4 + x \right) + \frac{x^5}{12} - \frac{x^2}{12}
\]

\[
= \frac{1}{4} x(x - 1).
\]

This can be confirmed using the standard solution form \( u(x) = u_c(x) + u_p(x) \) where

\[
u_c(x) = c_1 x + \frac{c_2}{x^2}.
\]

It is easily verified that \( u_p = x^2/4 \). BCs require \( c_2 = 0 \) and \( c_1 = -1/4 \) which recovers (2.171).
2.9 Green’s functions for the wave equation with time harmonic forcing

Consider the wave equation in one spatial dimension,

\[ \frac{\partial^2 U}{\partial \xi^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} + F(\xi, t) \]

where \( c \) is the wavespeed and \( F(\xi, t) \) is some forcing term. This applies to a number of physical systems, e.g. transverse waves on a string. Suppose that the forcing is time harmonic \( F(\xi, t) = g(\xi)e^{-i\omega t} \) so that the waves are also time harmonic, i.e.

\[ U(\xi, t) = u_0(\xi)e^{-i\omega t} \]

and then we have

\[ u''_0(\xi) + k^2 u_0(\xi) = g(\xi) \]

where \( k = \omega/c \) is the wavenumber (dimensions \( 1/L \)) and \( \omega \) is the frequency (dimensions \( 1/T \)). Note that using a complex exponential is for convenience. It is much neater mathematically than using a sine or cosine function. After all analysis the idea is to simply take the real part as clearly in reality we need a real solution.

We can non-dimensionalize (scale) \( \xi \) on the wavenumber, introducing \( x = k\xi \) and \( u(x) = ku_0 \) so that

\[ u''(x) + u(x) = f(x) \quad (2.172) \]

where \( f(x) = \frac{1}{k}g(x/k) \).

We have already determined the existence and uniqueness properties of this problem for the BCs \( u(0) = u(L) = 0 \) in Example 2.9. This corresponds to the situation when the ends of the string are fixed - a common scenario!

Let us assume, for now, that \( L \neq n\pi \) so that there are no existence and uniqueness issues. We then know the solution to the problem, it is (combining and simplifying all of the terms in Example 2.9)

\[ u(x) = \int_x^L \frac{\sin(x_0 - L) \sin x}{\sin L} f(x_0) \, dx_0 + \int_0^x \frac{\sin(x - L) \sin x_0}{\sin L} f(x_0) \, dx_0 \quad (2.173) \]

and we note that we can therefore write this as

\[ u(x) = \int_0^L G(x, x_0)f(x_0) \, dx_0 \quad (2.174) \]

where

\[ G(x, x_0) = \frac{1}{\sin L} \begin{cases} 
\sin(x_0 - L) \sin x, & 0 \leq x \leq x_0, \\
\sin x_0 \sin(x - L), & x_0 \leq x \leq L 
\end{cases} \quad (2.175) \]

and we note that \( G(x, x_0) = G(x_0, x) \). In particular it is helpful to write

\[ G(x, x_0) = \frac{1}{\sin L} (\sin(x_0 - L) \sin xH(x_0 - x) + \sin x_0 \sin(x - L)H(x - x_0)) \]
In question 1 on Example Sheet 5 we show that this Green’s function can be derived directly by following the steps in the “explicit solution” rather than the method of variation of parameters above.

Let us consider a slightly different problem now: an infinite domain.

**Example 2.17** Consider a string of infinite length. What this means in reality is that it is so long that the end conditions are never important in the problem. What is the associated Green’s function for forcing at the origin? Since we are interested in physical problems, let us work in the physical domain so let us solve

\[ G''(x, 0) + k^2 G(x, 0) = \delta(x) \]

where \( k = \omega/c \) is the wavenumber defined above. This is subject to the usual continuity conditions at \( x = x_0 = 0 \). Since the string is of infinite extent we do not have any BCs! So what do we use as additional conditions?! Well we know that if we are forcing the string harmonically at the point \( x = 0 \) then waves must be moving away from that point to infinity. They clearly cannot be coming in from infinity. So this applies for \( x \rightarrow \pm \infty \) which gives us another two conditions.

It is convenient in this case to write our fundamental solutions as \( u_1(x) = \exp(ix) \) and \( u_2(x) = \exp(-ix) \). In order for the solution to be outgoing as \( x \pm \infty \) it is clear that we need to take

\[ G(x, 0) = \begin{cases} c_L(0) \exp(-ix), & x \leq 0, \\ c_R(0) \exp(ix), & x \geq 0. \end{cases} \quad (2.176) \]

We have retained the general form with argument \( x_0 = 0 \) here since we will generalize to arbitrary \( x_0 \) later on. Why did we choose this form? Well we have time dependence of \( \exp(-i\omega t) \) so that for \( x > 0 \), we have a solution of the form \( \exp(i(kx - \omega t)) \). That this corresponds to a wave moving in the positive \( x \) direction is clear since when we increase time, if we want to stay at the same point on the wave we have to increase \( x \). Similarly \( \exp(-i(kx + \omega t)) \) corresponds to a left propagating wave and so is valid for \( x < 0 \). Another way of saying this is to impose that

\[ G'(x, 0) - ikG(x, 0) \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \quad (2.177) \]

\[ G'(x, 0) + ikG(x, 0) \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty. \quad (2.178) \]

Continuity at \( x = 0 \) gives \( c_L(0) = c_R(0) \). And the jump condition on the derivative gives

\[ \left[ \frac{dG}{dx} \right]_{x=0}^{x=0^+} = ikc_R(0) + ikc_L(0) = 2ikc_L(0) = 1 \quad (2.179) \]

so that \( c_L(0) = 1/(2ik) \) and

\[ G(x, 0) = \frac{1}{2ik} \exp(ik|x|). \quad (2.180) \]
It is not difficult to show that (see question 2 of Example Sheet 4) for a general point of forcing \( x = x_0 \),

\[
G(x, x_0) = \frac{1}{2i \kappa} \exp(ik|x - x_0|).
\tag{2.181}
\]

Note that \( G(x, x_0) = G(x_0, x) \). This is interesting because this is not Hermitian symmetry and therefore this problem cannot be self-adjoint. But this is strange. Why not?! It looks like it should be! Well, this is subtle: note that the coefficients in the “boundary (radiation) conditions” (2.177), (2.178) are complex and this means that we are not guaranteed self-adjointness. See part (ii) of question 2 on Example Sheet 4. In part (iii) of that question we also consider the problem of waves on a semi-infinite string forced harmonically.

Why work with complex solutions \( \exp(\pm ikx) \) rather trigonometric functions \( \sin kx \) and \( \cos kx \)? We do this for waves problems mainly because it is convenient in terms of algebra: when we take products of exponentials we can combine terms additively in the exponent. A good example is the time harmonic dependence:

\[
\exp(ikx) \exp(-i\omega t) = \exp(i(kx - \omega t)).
\]

Of course at the end we want REAL solutions so we have to take the real part of whatever solution we obtain in practice.
2.10 The adjoint Green’s function

We derived the result (2.132) only for S-A problems and in these cases we note that the Green’s function is also symmetric. What if the problem is not S-A? Can we still define a Green’s function and if so, what does the solution form for \( u(x) \) look like? We will deal with this here, although we note that physically many of the equations we deal with are of the S-A form.

First we introduce the governing equation

\[
\mathcal{L}_x u = f(x) \tag{2.182}
\]

where now \( \mathcal{L}_x \) is not S-A. Homogeneous BCs \( \mathcal{B} \) accompany (2.182). We note that we can introduce the corresponding Green’s function in the usual manner

\[
\mathcal{L}_x G(x, x_0) = \delta(x - x_0) \tag{2.183}
\]

with equivalent homogeneous BCs to the BVP.

**Theorem 2.5** For non S-A problems with homogeneous BCs, the solution can still be written

\[
u(x) = \int_a^b G(x, x_0) f(x_0) \ dx_0 \tag{2.184}\]

The proof of this is easy. We can perform \( \mathcal{L}_x \) on both sides of (2.184) as we did in (2.128).

However, how is this consistent with what was derived above using Green’s identity, because there we had to use the symmetry properties of the Green’s function? In fact what we can do is use the adjoint operator and define the so-called **adjoint Green’s function** \( G^* \) which satisfies

\[
\mathcal{L}^*_x G^*(x, x_2) = \delta(x - x_0) \tag{2.185}
\]

where \( \mathcal{L}^* \) is the adjoint operator together with the necessary adjoint BCs \( \mathcal{B}^* \).

We use the definition of the adjoint operator,

\[
\langle v, \mathcal{L} u \rangle = \langle \mathcal{L}^* v, u \rangle \tag{2.186}
\]

Let us choose \( u = G(x, x_1) \) and \( v = G^*(x, x_2) \), noting that \( G \) satisfies the same BCs as the original problem whereas \( G^* \) satisfies the adjoint BCs (note also here that we have seen a few examples of cases where \( \mathcal{L}^* = \mathcal{L} \) but the adjoint BCs are different and therefore this would clearly give \( G^*(x, y) \neq G(x, y) \)). This all gives

\[
\langle G^*(x, x_2), \mathcal{L} G(x, x_1) \rangle = \langle \mathcal{L}^* G^*(x, x_2), G(x, x_1) \rangle. \tag{2.187}\]

Using the definitions of these functions we find that

\[
\int_a^b G^*(x, x_2) \delta(x - x_1) \ dx = \int_a^b \mathcal{L}^* G^*(x, x_2) G(x, x_1) \ dx \tag{2.188}\]
and the sifting property of the delta function gives

\[ G^*(x_1, x_2) = \int_a^b \mathcal{L}^* G^*(x, x_2) G(x, x_1) \, dx \]

\[ = \int_a^b \delta(x - x_2) G(x, x_1) \, dx \]

\[ = G(x_2, x_1) \quad (2.189) \]

So we have proved the following result:

**Theorem 2.6** For operators that are not fully S-A, the Green’s function is not Hermitian symmetric. But there is a symmetry relation relating the Green’s function and its adjoint:

\[ G(x, x_0) = G^*(x_0, x). \quad (2.190) \]

Next, choose \( u \) to satisfy the original inhomogeneous problem and \( v = G^* \) in (2.186) so that we obtain

\[ \langle G^*, \mathcal{L} u \rangle - \langle u, \mathcal{L}^* G^* \rangle = 0. \quad (2.191) \]

Once again, using the sifting property of the delta function and interchanging variables we obtain

\[ u(x) = \int_a^b G^*(x_0, x) f(x_0) \, dx_0. \quad (2.192) \]

Finally, we use Theorem 2.6 to get

\[ u(x) = \int_a^b G(x, x_0) f(x_0) \, dx_0. \quad (2.193) \]

What the theory just presented regarding the adjoint Green’s function tells us is that we do not have to ever worry about constructing the adjoint Green’s function! We can always just construct the Green’s function (if it exists), satisfying the same BCs as the original problem, even when it is not S-A and still write the solution in the form

\[ u(x) = \int_a^b G(x, x_0) f(x_0) \, dx_0. \]

This is convenient!

In the next section we will construct a Green’s function for a non S-A operator.
2.11 Green’s functions for non S-A BVPs

The theory in this section (in blue below) is non-examinable.

What you should take from this section is that for Non-SA problems, you can use the same explicit method to derive the Green’s function as we derived for regular S-L problems BUT ONLY WHEN BCs are NOT MIXED! I WILL NOT SET ANY PROBLEMS ON THE EXAM INVOLVING FINDING THE GREENS FUNCTION WITH MIXED BCs!

Above we showed explicitly how to determine the Green’s function when the problem is of regular S-L type. What if it is not of this type? Three different cases arise. These are:

(i) When the ODE is in S-A form, with $B^* \neq B$ (but BCs are not mixed)
(ii) When the ODE is not in S-A form with non-mixed BCs.
(iii) When any BC is of mixed type.

In case (i) we can follow the explicit approach described above for regular S-L problems with no problem.

In case (ii) we now show that we can transform the ODE into S-A form and hence there is no issue. One can approach the Green’s function construction in exactly the manner described for regular S-L problems above. We can force the ODE into S-A form as now explain.

Take the usual ODE
\[ p(x) \frac{d^2 G}{dx^2} + r(x) \frac{dG}{dx} + q(x)G = \delta(x - x_0). \] (2.194)

Divide by $p(x)$, generate the integrating factor
\[ I(x) = \exp \left( \int \frac{r(x)}{p(x)} \, dx \right) \]
and then multiply through by this to get
\[ I(x) \frac{d^2 G}{dx^2} + \frac{r(x)}{p(x)} I(x) \frac{dG}{dx} + q(x)I(x)G = \frac{I(x)}{p(x)} \delta(x - x_0). \]

and now re-write the left hand side in the form
\[ \frac{d}{dx} \left( I(x) \frac{dG}{dx} \right) + q(x)I(x)G = \frac{I(x)}{p(x)} \delta(x - x_0). \] (2.195)

It may not be beneficial to do this in order to solve the problem. Indeed solving (2.194) may be easier (but they must give the same solution!). But we can now integrate (2.195) between $x = x_0^-$ and $x = x_0^+$ to get the jump condition as before
\[ \left[ \frac{dG}{dx} \right]_{x_0^-}^{x_0^+} = \frac{1}{p(x_0)}. \] (2.196)

Hence we can in fact follow exactly the same procedure as for regular S-L problems in order to generate the Green’s function. The only difference is that the Green’s function will not be Hermitian symmetric. Instead we have the alternative symmetry relation $G(x_0, x) = G^*(x, x_0)$ as proven above.
Example 2.18 Solve the BVP consisting of the ODE

\[ x^2 u'' + 4xu' + 2u = f(x) \]

and the BCs \( u(1) = u(2) = 0 \) by finding the Green’s function and write down the explicit solution in the case when \( f(x) = x \). Confirm that this is what one would expect by solving via direct methods for this specific \( f(x) \).

Fundamental solutions to the homogeneous problem are given by solving

\[ x^2 u'' + 4xu + 2u = 0 \]

which is an Euler equation so we seek solutions of the form \( u(x) = x^m \) which gives

\[ m^2 + 3m + 2 = (m + 2)(m + 1) = 0 \]

so that \( u(x) = x^{-2} \) and \( u(x) = x^{-1} \).

Combination of these solutions satisfying the left and right BCs are

\[ u_L(x) = \frac{1}{x} - \frac{1}{x^2} \quad \text{and} \quad u_R(x) = \frac{1}{x} - \frac{2}{x^2} \]  \hspace{1cm} (2.197)

The associated Wronskian is (check this!)

\[ W(x_0) = u_L(x_0)u_R'(x_0) - u_R(x_0)u_L'(x_0) \]

so that \( p(x_0)W(x_0) = 1/x_0^2 \). Therefore

\[ c_L(x_0) = \frac{u_R(x_0)}{p(x_0)W(x_0)} = x_0 - 2, \quad c_R(x_0) = \frac{u_L(x_0)}{p(x_0)W(x_0)} = x_0 - 1. \]  \hspace{1cm} (2.198)

and the Green’s function is

\[ G(x, x_0) = \begin{cases} 
  c_L(x_0)u_L(x), & 1 \leq x \leq x_0, \\
  c_R(x_0)u_R(x), & x_0 \leq x \leq 2.
\end{cases} \]

\[ = \begin{cases} 
  \frac{1}{x^2}(x_0 - 2)(x - 1), & 1 \leq x \leq x_0, \\
  \frac{1}{x^2}(x_0 - 1)(x - 2), & x_0 \leq x \leq 2.
\end{cases} \]

\[ = \frac{1}{x^2}(x_0 - 2)(x - 1)H(x_0 - x) + \frac{1}{x^2}(x_0 - 1)(x - 2)H(x - x_0). \]

The solution of the problem is

\[ u(x) = \int_1^2 f(x_0)G(x, x_0) \, dx_0. \]

With \( f(x) = x \), carrying out the integrations we obtain (check this!)

\[ u(x) = \frac{1}{x^2} - \frac{7x}{6} + \frac{x}{6} \]

In question 4 on Example Sheet 5 you are asked to show that the same result would have been obtained if you had used the adjoint Green’s function.
THE REST OF SECTION 2.11 IS NON-EXAMINABLE.

If BCs are of mixed type (i.e. (iii) above), then strictly there is no “left” or “right” BC to satisfy. What this means is that we cannot solve the problem in the same manner as the explicit method above. In particular it means that the \( x \) and \( x_0 \) dependence in the Green’s function is NOT separable.

Once we have determined fundamental solutions \( u_1 \) and \( u_2 \) what we must do is to pose a solution in the form

\[
G(x, x_0) = \begin{cases} 
  c_L(x_0)u_1(x) + d_L(x_0)u_2(x), & a \leq x \leq x_0, \\
  c_R(x_0)u_1(x) + d_R(x_0)u_2(x), & x_0 \leq x \leq b 
\end{cases}
\]

and then determine \( c_L, d_L, c_R, d_R \) from the two boundary conditions and continuity conditions imposed at \( x = x_0 \).

Let us illustrate with an example.

Example 2.19 Determine the Greens function for the ODE

\[
u''(x) + 3u'(x) + 2u(x) = 0
\]

subject to the mixed BCs \( u(0) = u'(1) \) and \( u(1) = 2u(0) + u'(0) \).

Both BCs are mixed so there is no “left” or “right” BC. However we can easily solve the homogeneous ODE with

\[
u_1(x) = e^{-x}, \quad u_2(x) = e^{-2x}
\]

so let us take the Green’s function in the form above:

\[
G(x, x_0) = \begin{cases} 
  c_L(x_0)e^{-x} + d_L(x_0)e^{-2x}, & 0 \leq x \leq x_0, \\
  c_R(x_0)e^{-x} + d_R(x_0)e^{-2x}, & x_0 \leq x \leq 1
\end{cases}
\]

Imposing the BCs leads to the conditions

\[
c_L + d_L = -c_R \frac{1}{e} - 2d_R \frac{e}{e^2},
\]

\[
c_R + d_R = c_L
\]

together with the standard continuity conditions at \( x = x_0 \) (which give an addition two conditions) we can then determine \( c_L, d_L, c_R, d_R \). (see question 8 on Example Sheet 5).

2.12 Inhomogeneous boundary conditions

In general we would like to solve problems which are not restricted to boundary conditions that are homogeneous. There are two approaches to solving such problems.

Consider the following problem

\[
\mathcal{L}U = f(x)
\]
subject to \(U(a) = \alpha, U(b) = \beta\). Since the problem is linear we can decompose the solution in the form

\[
U(x) = u(x) + v(x)
\]  

(2.202)

where \(u\) and \(v\) satisfy

\[
\mathcal{L}u = f(x), \quad \mathcal{L}v = 0
\]  

(2.203)

subject to \(u(a) = 0, u(b) = 0\) and \(v(a) = \alpha, v(b) = \beta\). The problem for \(v\) is simply to find a linear combination of the fundamental solutions such that the boundary conditions are met. However note that from Example (2.8) we are not always guaranteed that such a solution will exist!

The problem for \(u\) is equivalent to the homogeneous boundary condition problems above with associated Green’s function that also satisfies homogeneous BCs. Thus if both \(v(x)\) and the Green’s function exists, the solution will be

\[
U(x) = \int_{a}^{b} G(x, x_0) f(x_0) \, dx_0 + v(x)
\]  

(2.204)

Alternatively we can derive the solution directly via the Green’s function and application of Lagrange’s identity. For conciseness let us consider the fully self-adjoint case. Then

\[
\int_{a}^{b} \mathcal{G} U - U \mathcal{G} \, dx = \left[ G \frac{dU}{dx} - U \frac{dG}{dx} \right]_{a}^{b} = U(a) \frac{dG}{dx}(a, x_0) - U(b) \frac{dG}{dx}(b, x_0)
\]  

\[
= \alpha \frac{dG}{dx}(a, x_0) - \beta \frac{dG}{dx}(b, x_0)
\]  

(2.205)

Therefore simplifying the left hand side, the solution is

\[
U(x) = \int_{a}^{b} G(x, x_0) f(x_0) \, dx_0 + \beta \frac{dG}{dx}(b, x) - \alpha \frac{dG}{dx}(a, x).
\]  

(2.206)

The two solutions (2.204) and (2.206) are equivalent of course.

An example associated with non-homogeneous boundary conditions can be found in question 9 on Example Sheet 5.
THIS LAST SECTION IS NOT EXAMINABLE. BUT:
You should ensure you know what goes wrong when there is a zero eigenvalue, i.e. the normal green’s function cannot be defined and therefore you need a “modified” Green’s function. But you won’t be asked to construct any or know the theory of them, etc.

2.13 Existence of a zero eigenvalue - modified Green’s functions

As is clear from the eigenfunction expansion of the Green’s function in (2.110), there is clearly a problem when there exists a zero eigenvalue. In this instance the Green’s function does not exist! As we know, a zero eigenvalue corresponds to a non-trivial solution of the homogeneous BVP, and this exists for the fixed ends string problem above, when $L = n\pi$ so let us consider that problem. Try to determine the Green’s function using the method above - it will not work!

In this case as we know from the Fredholm Alternative, there will only exist solutions to the inhomogeneous BVP if

$$
\int_0^L \sin x_0 f(x_0) \, dx_0 = 0,
$$

so we assume that this holds. If $n$ is even, any even function $f(x_0)$ will ensure this.

The reason that the Green’s function does not exist is because, once again by the Fredholm Alternative, for its existence (it is itself defined by an inhomogeneous BVP) we require

$$
\int_0^L \sin x_0 \delta(x - x_0) \, dx_0 = \sin x = 0
$$

for all $x \in [0, L]$ but this clearly does not hold.

Therefore instead of using that Green’s function as defined classically, what we should do is introduce a different comparison problem. We define the so-called modified Green’s function $G_m(x, x_0)$ via the governing equation

$$
L G_m(x, x_0) = \delta(x - x_0) + c \phi(x)
$$

where $c \in \mathbb{R}$ and $\phi(x)$ is the non-trivial eigenfunction corresponding to the zero eigenvalue (here it is $\phi(x) = \sin x$). All other conditions on the Green’s function remain the same.

In the general case then, we must choose $c$ so that the right hand side is orthogonal to the eigenfunction corresponding to the zero eigenvalue. I.e. here we choose

$$
\int_a^b (\delta(z - x_0) + c\phi(z))\phi(z) \, dz = 0
$$

so that

$$
c = -\frac{\phi(x_0)}{\int_a^b \phi^2(z) \, dz}
$$

where $x_0$ is the location of the source. Therefore the modified Green’s function $G_m(x, x_0)$ is defined by the equation

$$
L G_m(x, x_0) = \delta(x - x_0) - \frac{\phi(x)\phi(x_0)}{\int_a^b \phi^2(z) \, dz}.
$$
subject to the usual homogeneous BCs. Unfortunately since the right hand side is orthogonal to \(\phi(x)\), by the Fredholm Alternative there are an infinite number of solutions so that the modified Green’s function is not uniquely defined.

It transpires that the particular solution of the BVP can be written as usual in terms of the modified Green’s function \(G_m(x,x_0)\) (which can be chosen to be symmetric) in the form

\[
u(x) = \int_a^b G_m(x,x_0)f(x_0) \, dx_0.
\]

**Example 2.20** Derive the modified Green’s function for the problem

\[
u''(x) = f(x)
\]

subject to \(u'(0) = 0, u'(L) = 0\).

As we have discussed above, a constant \(c\) is a homogeneous solution (an eigenfunction corresponding to a zero eigenvalue). We note that by the Fredholm Alternative, for a solution to exist therefore we must have

\[
\int_0^L f(x) \, dx = 0.
\]

Let us assume that we have such an \(f(x)\). In that case the modified Green’s function satisfies

\[
\frac{d^2 G_m}{dx^2} = \delta(x-x_0) + c.
\]

subject to \(G'_m(0,x_0) = 0, G'_m(L,x_0) = 0\). For a modified Green’s function to exist, again by the Fredholm Alternative the right hand side has to be orthogonal to a constant, i.e.

\[
\int_0^L \delta(x-x_0) + c \, dx = 0
\]

so that \(c = -1/L\).

Thus for \(x \neq y\), we must have

\[
\frac{d^2 G_m}{dx^2} = -\frac{1}{L}.
\]

By direct integration,

\[
\frac{d G_m}{dx} = \begin{cases} \frac{-x}{L} + c_1, & 0 \leq x \leq x_0, \\ \frac{-x}{L} + c_2, & x_0 \leq x \leq L. \end{cases}
\]

If we choose \(c_1\) and \(c_2\) to satisfy the BCs at \(x = 0, L\) we find \(c_1 = 0, c_2 = 1\). The jump condition on \(dG_m/dx\) is already satisfied. Integrating once again we find

\[
G_m(x,x_0) = \begin{cases} \frac{-x^2}{2L} + d_1, & 0 \leq x \leq x_0, \\ \frac{-x^2}{2L} + x + d_2, & x_0 \leq x \leq L. \end{cases}
\]
Imposing continuity of the Green’s function at $x = x_0$ we get $d_1 = x_0 + d_2$. So we find

$$G_m(x, x_0) = \begin{cases} 
-\frac{x^2}{2L} + x_0 + d_2, & 0 \leq x \leq x_0, \\
\frac{-x}{2L} + x + d_2, & x_0 \leq x \leq L.
\end{cases}$$

which illustrates that it is not unique. Imposing symmetry of the Green’s function, i.e. we find that (see question ?? on Example Sheet 5)

$$d_2 = -\frac{1}{L} \frac{x_0^2}{2} + \beta$$

where $\beta$ is a constant, independent of $x_0$.

Therefore we have,

$$G_m(x, x_0) = \begin{cases} 
-\frac{x^2}{2L} + x_0 + \beta, & 0 \leq x \leq x_0, \\
\frac{-x}{2L} + x + \beta, & x_0 \leq x \leq L.
\end{cases}$$

and a solution of the BVP is

$$u(x) = \int_0^L G_m(x, x_0)f(x_0) \, dx_0.$$ 

since $f(x_0)$ is orthogonal to a constant, we can take $\beta = 0$ without loss of generality.
2.14 Revision checklist

The following is a guide to what you should know. Read each point and ask yourself if you understand what it means! Also, remember that associated theory from the relevant sections is examinable.

- For constant coefficient and Euler ODEs:
  - be able to find the complementary function \( u_c \) and
  - be able to find the particular solution \( u_p \) by method of undetermined coefficients and variation of parameters.

- Use integration by parts with inner products to derive the adjoint operator and BCs.

- Know Lagrange’s and Green’s identities and be able to use them.

- Be able to identify self adjoint operators - this requires both \( L^* = L \) and \( B^* = B \).

- Be able to identify a Sturm-Liouville problem and know the difference between regular and singular Sturm-Liouville problems.

- Know the 5 theorems associated with regular S-L problems and be able to see how they relate to model problems.

- Understand the Fredholm Alternative theorem and be able to apply it to determine if solutions to inhomogeneous ODEs (with homogeneous BCs) exist and are unique.

- Be able to derive the Green’s function via eigenfunction expansions

- Understand some basic properties of the Dirac delta function and its relationship to the Green’s function

- Understand the conditions that define the Green’s function (governing BVP, conditions at \( x = x_0 \))

- Be able to derive the Green’s function for regular S-L problems by variation of parameters (i.e. applying v.o.p. to the original ODE), and the direct approach (explicit solution - section 2.8.7)

- Be able to derive Green’s functions for the wave equation at fixed frequency (time harmonic)

- Understand the problem that arises if a zero eigenvalue exists, understand its relationship with the Fredholm Alternative and the fact that a Green’s function does not exist in this case

- Be able to define and derive the adjoint Green’s function and its relationship to the Green’s function

- Be able to derive Green’s functions for non S-L problems (WHEN BCs are NOT MIXED).

- Understand linear superposition in order to derive solutions to inhomogeneous ODEs with inhomogeneous BCs