Section 3

Integral Equations

Integral Operators and Linear Integral Equations

As we saw in Section 1 on operator notation, we work with functions defined in some suitable function space. For example, \( f(x), g(x) \) may live in the space of continuous real-valued functions on \([a, b]\), i.e. \( C(a, b) \). We also saw that it is possible to define integral as well as differential operators acting on functions. Theorem 2.8 is an example of an integral operator:

\[
    u(x) = \int_{a}^{b} G(x, y) f(y) \, dy,
\]

where \( G(x, y) \) is a Green’s function.

**Definition 3.1:** An *integral operator*, \( K \), is an integral of the form

\[
    \int_{a}^{b} K(x, y) f(y) \, dy,
\]

where \( K(x, y) \) is a given (real-valued) function of two variables, called the kernel.

The integral equation

\[
    K f = g
\]

maps a function \( f \) to a new function \( g \) in the interval \([a, b] \), e.g.

\[
    K : C [a, b] \to C [a, b], \quad K : f \mapsto g.
\]

**Theorem 3.2:** The integral operator \( K \) is linear

\[
    K (\alpha f + \beta g) = \alpha K f + \beta K g.
\]

**Example 1:** The Laplace Transform is an integral operator

\[
    \mathcal{L} f = \int_{0}^{\infty} \exp(-sx) f(x) \, dx = \tilde{f}(s),
\]

with kernel \( K(s, x) = K(x, s) = \exp(-sx) \). Let us recap its important properties.

**Theorem 3.3:**

(i) \( \mathcal{L} \left( \frac{df}{dx} \right) = s\tilde{f}(s) - f(0), \)

(ii) \( \mathcal{L} \left( \frac{d^2f}{dx^2} \right) = s^2\tilde{f}(s) - sf(0) - f'(0), \)

(iii) \( \mathcal{L} \left( \int_{0}^{x} f(y) \, dy \right) = \frac{1}{s} \tilde{f}(s). \)

(iv) Convolution: let

\[
    f(x) \ast g(x) = \int_{0}^{x} f(y) g(x - y) \, dy
\]

then

\[
    \mathcal{L} (f(x) \ast g(x)) = \tilde{f}(s) \tilde{g}(s).
\]
Recall that a differential equation is an equation containing an unknown function under a differential operator. Hence, we have the same for an integral operator.

**Definition 3.4:** An integral equation is an equation containing an unknown function under an integral operator.

**Definition 3.5:**
(a) A linear Fredholm integral equation of the first kind has the form
\[ Kf = g, \quad \int_a^b K(x, y) f(y) \, dy = g(x); \]
(b) A linear Fredholm integral equation of the second kind has the form
\[ f - \lambda Kf = g, \quad f(x) - \lambda \int_a^b K(x, y) f(y) \, dy = g(x), \]

where the kernel \( K(x, y) \) and forcing (or inhomogeneous) term \( g(x) \) are known functions and \( f(x) \) is the unknown function. Also, \( \lambda \in \mathbb{R} \) or \( \mathbb{C} \) is a parameter.

**Definition 3.6:** If \( g(x) \equiv 0 \) the integral equation is called homogeneous and a value of \( \lambda \) for which \( \lambda K \phi = \phi \) possesses a non-trivial solution \( \phi \) is called an eigenvalue of \( K \) corresponding to the eigenfunction \( \phi \).

**Note:** \( \lambda \) is the reciprocal of what we have previously called an eigenvalue.

**Definition 3.7** Volterra integral equations of the first and second kind take the forms
(a) \[ \int_a^x K(x, y) f(y) \, dy = g(x), \]
(b) \[ f(x) - \lambda \int_a^x K(x, y) f(y) \, dy = g(x), \]
respectively.

**Note:** Volterra equations may be considered a special case of Fredholm equations. Suppose \( a \leq y \leq b \) and put
\[ K_1(x, y) = \begin{cases} K(x, y) & \text{for } a \leq y \leq x, \\ 0 & \text{for } x < y \leq b, \end{cases} \]
then
\[ \int_a^b K_1(x, y) f(y) \, dy = \left( \int_a^x + \int_x^b \right) K_1(x, y) f(y) \, dy = \int_a^x K(x, y) f(y) \, dy + 0. \]
Conversion of ODEs to integral equations

Example 2: Boundary value problems
Let
\[ \mathcal{L} u(x) = \lambda \rho(x) u(x) + g(x) \]
where \( \mathcal{L} \) is a linear ODE operator as in Section 1 (of at least second order), \( \rho(x) > 0 \) is continuous, \( g(x) \) is piecewise continuous and the unknown \( u(x) \) is subject to homogeneous boundary conditions at \( x = a, b \).
Suppose \( \mathcal{L} \) has a known Green’s function under the given boundary conditions, \( G(x, y) \) say, then applying Theorem 2.10,

\[
\begin{align*}
 u(x) &= \int_a^b G(x, y) [\lambda \rho(y) u(y) + g(y)] \, dy \\
 &= \lambda \int_a^b G(x, y) \rho(y) u(y) \, dy + \int_a^b G(x, y) g(y) \, dy \\
 &= \lambda \int_a^b G(x, y) \rho(y) u(y) \, dy + h(x).
\end{align*}
\]

The latter is a Fredholm integral equation of the second kind with kernel \( G(x, y) \rho(y) \) and forcing term
\[ h(x) = \int_a^b G(x, y) g(y) \, dy. \]

The ODE and integral equation are equivalent: solving the ODE subject to the BCs is equivalent to solving the integral equation.

Example 3: Initial value problems
Let
\[ u''(x) + a(x) u'(x) + b(x) u(x) = g(x), \]
where \( u(0) = \alpha \) and \( u'(0) = \beta \) and \( a(x), b(x) \) and \( g(x) \) are known functions.
Change dummy variable from \( x \) to \( y \) and then Integrate with respect to \( y \) from 0 to \( z \):
\[
\int_0^z u''(y) \, dy + \int_0^z a(y) u'(y) \, dy + \int_0^z b(y) u(y) \, dy = \int_0^z g(y) \, dy
\]
and using integration by parts, with \( u = a, v' = u' \), on the second term on the left hand side
\[
[u'(y)]_0^z + [a(y) u(y)]_0^z - \int_0^z a'(y) u(y) \, dy + \int_0^z b(y) u(y) \, dy = \int_0^z g(y) \, dy,
\]
yields
\[
u'(z) - \beta + a(z) u(z) - a(0) \alpha - \int_0^z [a'(y) - b(y)] u(y) \, dy = \int_0^z g(y) \, dy.
\]
Integrating now with respect to \( z \) over 0 to \( x \) gives
\[
[u(z)]_0^x - \beta x + \int_0^x a(z) u(z) \, dz - a(0) \alpha x \quad (3.1)
- \int_0^x \int_0^z [a'(y) - b(y)] u(y) \, dydz
= \int_0^x \int_0^z g(y) \, dydz.
\]
This can be simplified by appealing to the following theorem.

**Theorem 3.8:** For any \( f(x) \),

\[
\int_{0}^{x} \int_{0}^{z} f(y) \, dy \, dz = \int_{0}^{x} \left( x - y \right) f(y) \, dy.
\]

**Proof:** Interchanging the order of integration over the triangular domain in the \( yz \)-plane reveals that the integral equals

\[
\int_{0}^{x} \int_{0}^{z} f(y) \, dy \, dz = \int_{0}^{x} f(y) \int_{y}^{x} dz \, dy
\]

which is trivially integrated over \( z \) to give

\[
\int_{0}^{x} (x - y) f(y) \, dy
\]

as required. Alternatively, integrating the repeated integral by parts with \( u(z) = \int_{0}^{z} f(y) \, dy \), \( v'(z) = 1 \), i.e. \( u'(z) = f(z) \), \( v(z) = z \), gives

\[
\int_{0}^{x} \int_{0}^{z} f(y) \, dy \, dz = \left[ z \int_{0}^{z} f(y) \, dy \right]_{z=0}^{z=x} - \int_{0}^{x} zf(z) \, dz
\]

\[
= x \int_{0}^{x} f(y) \, dy - \int_{0}^{x} zf(z) \, dz
\]

\[
= \int_{0}^{x} xf(y) \, dy - \int_{0}^{x} yf(y) \, dy.
\]

Returning to equation (3.1), it may now be written as

\[
\begin{align*}
\int_{0}^{x} & \left( x - y \right) g(y) \, dy + [\beta + a(0) \alpha] x + \alpha,
\end{align*}
\]

Thus, \( u(x) + \int_{0}^{x} \{ a(y) - (x - y) [a'(y) - b(y)] \} u(y) \, dy \)

This is a Volterra integral equation of the second kind.

**Notes:**

(1) BVPs correspond to Fredholm equations.
(2) Initial value problems (IVPs) correspond to Volterra equations.
(3) The integral equation formulation implicitly contains the boundary/initial conditions; in the ODE they are imposed separately.
Example 4

Write the Initial Value Problem

\[ u''(y) + yu'(y) + 2u(y) = 0 \]

subject to \( u(0) = \alpha, u'(0) = \beta \) as a Volterra integral equation.

Integrate with respect to \( y \) between 0 and \( z \):

\[ u'(z) - \beta + \int_0^z yu'(y)dy + 2 \int_0^z u(y)dy = 0 \]

Integrate second term by parts:

\[ \int_0^z yu'(y)dy = [yu(y)]_0^z - \int_0^z u(y)dy, \]

\[ = zu(z) - \int_0^z y(y)dy \]

and substitute back to get

\[ u'(z) - \beta + zu(z) + \int_0^z u(y)dy = 0. \]

Integrate again, with respect to \( z \) between 0 and \( x \):

\[ u(x) - \alpha - \beta x + \int_0^x zu(z)dz + \int_0^x \int_0^z u(y)dydz = 0. \]

Change dummy variable in first integral term to \( y \) and use Theorem 3.8 in last term to get

\[ u(x) - \alpha - \beta x + \int_0^x yu(y)dy + \int_0^x (x - y)u(y) = 0 \]

Simplification gives

\[ u(x) + x \int_0^x u(y)dy = \alpha + \beta x. \]
Fredholm Equations with Degenerate Kernels

We have seen that a Fredholm integral equation of the second kind is defined as

\[ f(x) = \lambda \int_a^b K(x, y) f(y) \, dy + g(x). \tag{3.2} \]

**Definition 3.9:** The kernel \( K(x, y) \) is said to be **degenerate (separable)** if it can be written as a sum of terms, each being a product of a function of \( x \) and a function of \( y \). Thus,

\[ K(x, y) = \sum_{j=1}^{n} u_j(x) v_j(y) = u^T(x) v(y) = u(x) \cdot v(y) = \langle u, v \rangle, \tag{3.3} \]

where the latter notation is the inner product for finite vector spaces (i.e. the dot product).

Equation (3.2) may be solved by reduction to a set of simultaneous linear algebraic equations as we shall now show. Substituting (3.3) into (3.2) gives

\[
 f(x) = \lambda \int_a^b \left[ \sum_{j=1}^{n} u_j(x) v_j(y) \right] f(y) \, dy + g(x)
\]

\[
 = \lambda \sum_{j=1}^{n} \left[ u_j(x) \int_a^b v_j(y) f(y) \, dy \right] + g(x)
\]

and letting

\[ c_j = \int_a^b v_j(y) f(y) \, dy = \langle v_j, f \rangle, \tag{3.4} \]

then

\[ f(x) = \lambda \sum_{j=1}^{n} c_j u_j(x) + g(x). \tag{3.5} \]

For this class of kernel, it is sufficient to find the \( c_j \) in order to obtain the solution to the integral equation. Eliminating \( f \) between equations (3.4) and (3.5) (i.e. take inner product of both sides with \( v_i \)) gives

\[ c_i = \int_a^b v_i(y) \left[ \lambda \sum_{j=1}^{n} c_j u_j(y) + g(y) \right] dy, \]

or interchanging the summation and integration,

\[ c_i = \lambda \sum_{j=1}^{n} c_j \int_a^b v_i(y) u_j(y) \, dy + \int_a^b v_i(y) g(y) \, dy. \tag{3.6} \]

Writing

\[ a_{ij} = \int_a^b v_i(y) u_j(y) \, dy = \langle v_i, u_j \rangle, \tag{3.7} \]

and

\[ g_i = \int_a^b v_i(y) g(y) \, dy = \langle v_i, g \rangle, \tag{3.8} \]
then (3.6) becomes
\[ c_i = \lambda \sum_{j=1}^{n} a_{ij} c_j + g_i. \] (3.9)

By defining the matrices
\[
A = (a_{ij}), \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}
\]
this equation may be written in matrix notation as
\[ c = \lambda A c + g \]
i.e.
\[ (I - \lambda A) c = g \] (3.10)
where \( I \) is the identity. This is just a simple linear system of equations for \( c \). We therefore need to understand how we solve the canonical system \( Ax = b \) where \( A \) is a given matrix, \( b \) is the given forcing vector and \( x \) is the vector to be determined. Let’s state an important theorem from Linear Algebra:

**Theorem 3.10: (Fredholm Alternative)**

Consider the linear system
\[ Ax = b \] (3.11)
where \( A \) is an \( n \times n \) matrix, \( x \) is an unknown \( n \times 1 \) column vector, and \( b \) is a specified \( n \times 1 \) column vector.

We also introduce the related (adjoint) homogeneous problem
\[ A^T \hat{x} = 0 \] (3.12)
with \( p = n - \text{rank}(A) \) non-trivial linearly independent solutions
\[ \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p. \]

[Reminder, \( \text{rank}(A) \) is the number of linearly independent rows (or columns) of the matrix \( A \).]

Then the following alternatives hold:

*either*

(i) \( \text{Det}A \neq 0 \), so that there exists a unique solution to (3.11) given by \( x = A^{-1}b \) for each given \( b \). (And \( b = 0 \Rightarrow x = 0 \))

or

(ii) \( \text{Det}A = 0 \) and then

(a) If \( b \) is such that \( \langle b, \hat{x}_j \rangle = 0 \) for all \( j \) then there are infinitely many solutions to equation (3.11).

(b) If \( b \) is such that \( \langle b, \hat{x}_j \rangle \neq 0 \) for any \( j \) then there is no solution to equation (3.11).
In the case of (ii)(a), then there are infinitely many solutions because the theorem states that we can find a particular solution $x_{PS}$ and furthermore, the homogeneous system
\[ Ax = 0 \] (3.13)
has $p = n - \text{rank}(A) > 0$ non-trivial linearly independent solutions $x_1, x_2, \ldots, x_p$.
so that there are infinitely many solutions because we can write
\[ x = x_{PS} + \sum_{j=1}^{p} \alpha_j \hat{x}_j \]
where $\alpha_j$ are arbitrary constants (and hence there are infinitely many solutions).
No proof of this theorem is given.

To illustrate this theorem consider the following simple $2 \times 2$ matrix example:

**Example 5**

Determine the solution structure of the linear system $Ax = b$ when

\begin{align*}
\text{(I)} & \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\
\text{(II)} & \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\end{align*} (3.14)

and in the case of (II) when
\[ b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

(I) Since $\text{Det}(A) = 1 \neq 0$ the solution exists for any $b$, given by $x = A^{-1}b$.

(II) Here $\text{Det}(A) = 0$ so we have to consider solutions to the adjoint homogeneous system, i.e.

\[ A^T \hat{x} = 0 \] (3.16)

i.e.
\[ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \hat{x} = 0. \] (3.17)

This has the 1 non-trivial linearly independent solution $\hat{x}_1 = (2 - 1)^T$. It is clear that there should be 1 such solution, i.e. $p = n - \text{rank}(A) = 2 - 1 = 1$.

Note also that the homogeneous system
\[ A\hat{x} = 0 \] (3.18)

i.e.
\[ \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \hat{x} = 0 \] (3.19)

has the 1 non-trivial linearly independent solution $x_1 = (1 - 1)^T$. If solutions do exist they will therefore have the form $x = x_{PS} + \alpha_1 x_1$.

A solution to the problem $Ax = b$ will exist if $\hat{x}_1 \cdot b = 0$. This condition does hold for $b = (1, 2)^T$ and so the theorem predicts that a solution will exist. Indeed it does, note that $x_{PS} = (1/2, 1/2)^T$ and so $x = x_{PS} + \alpha_1 x_1$ is the infinite set of solutions.

The orthogonality condition does not hold for $b = (1, 1)^T$ and so the theorem predicts that a solution will not exist. This is clear from looking at the system.
Now let us apply the Fredholm Alternative theorem to equation (3.10) in order to solve the problem of degenerate kernels in general. 

**Case (i)** if

$$\det (I - \lambda A) \neq 0 \quad (3.20)$$

then the Fredholm Alternative theorem tells us that (3.10) has a unique solution for \( c \):

$$c = (I - \lambda A)^{-1} g. \quad (3.21)$$

Hence (3.2), with degenerate kernel (3.3), has the solution (3.5):

$$f(x) = \lambda \sum_{i=1}^{n} c_i u_i(x) + g(x) = \lambda (u(x))^T c + g(x)$$

or from (3.21)

$$f(x) = \lambda (u(x))^T (I - \lambda A)^{-1} g + g(x),$$

which may be expressed, from (3.8), as

$$f(x) = \lambda \int_{a}^{b} [(u(x))^T (I - \lambda A)^{-1} v(y)] g(y) dy + g(x).$$

**Definition 3.11:** The **resolvent kernel** \( R(\lambda, x, y) \) is such that the integral representation for the solution

$$f(x) = \lambda \int_{a}^{b} R(\lambda, x, y) g(y) dy + g(x)$$

holds.

**Theorem 3.12:** For a degenerate kernel, the resolvent kernel is given by

$$R(\lambda, x, y) = (u(x))^T (I - \lambda A)^{-1} v(y).$$

Case (i) covered the simple case when there is a unique solution. Let us now concern ourselves with the case when the determinant of the matrix on the left hand side of the linear system is zero. **Case (ii)** suppose

$$\det (I - \lambda A) = 0, \quad (3.22)$$

and that the homogeneous equation

$$(I - \lambda A) c = 0 \quad (3.23)$$

has \( p \) non-trivial linearly independent solutions

$$c^1, c^2, \ldots, c^p.$$

Then, the homogeneous form of the integral equation (3.2), i.e.

$$f(x) = \lambda \int_{a}^{b} K(x, y) f(y) dy, \quad (3.24)$$
with degenerate kernel (3.3), has \( p \) solutions, from (3.5):

\[
f^j (x) = \lambda \sum_{i=1}^{n} c^j_i u_i (x)
\]

with \( j = 1, 2, \ldots, p \).

Turning to the inhomogenous equation, (3.10), it has a solution if and only if the forcing term \( g \) is orthogonal to every solution of

\[
(I - \lambda A)^T h = 0 \quad \text{i.e.} \quad h^T g = 0
\]

or

\[
\sum_{i=1}^{n} h_i g_i = 0.
\]

Hence (3.8) yields

\[
\sum_{i=1}^{n} h_i \int_{a}^{b} v_i (y) g (y) \, dy = 0
\]

which is equivalent to

\[
\int_{a}^{b} \left( \sum_{i=1}^{n} h_i v_i (y) \right) g (y) \, dy = 0.
\]

Thus, writing

\[
h (y) = \sum_{i=1}^{n} h_i v_i (y)
\]

then

\[
\int_{a}^{b} h (y) g (y) \, dy = 0,
\]

which means that \( g (x) \) must be orthogonal to \( h (x) \) on \([a, b]\).

Let us explore the function \( h(x) \) a little; we start be expressing (3.26) as

\[
h_i - \lambda \sum_{j=1}^{n} a_{ji} h_j = 0.
\]

Without loss of generality assume that all the \( v_i (x) \) in (3.3) are linearly independent (since if one is dependent on the others, eliminate it and obtain a separable kernel with \( n \) replaced by \( n - 1 \)). Multiply the \( i \)th equation in (3.26) by \( v_i (x) \) and sum over all \( i \) from 1 to \( n \):

\[
\sum_{i=1}^{n} h_i v_i (x) - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ji} h_j v_i (x) = 0,
\]

i.e. from (3.7)

\[
\sum_{i=1}^{n} h_i v_i (x) - \lambda \int_{a}^{b} \sum_{i=1}^{n} \sum_{j=1}^{n} h_j v_j (y) u_i (y) v_i (x) \, dy = 0.
\]

Using (3.27) and (3.3) we see that this reduces to the integral equation:

\[
h (x) - \lambda \int_{a}^{b} K (y, x) h (y) \, dy = 0.
\]

(3.28)
Note that this is the homogeneous form of the transpose of the Fredholm integral equation (3.2), i.e. it has no forcing term on the right hand side and the kernel is written $K(y, x)$ rather than $K(x, y)$.

In conclusion for Case (ii), the integral equation (3.2) with a separable kernel of the form (3.3) has a solution if and only if $g(x)$ is orthogonal to every solution $h(x)$ of the homogeneous equation (3.28). The general solution is then

$$f(x) = f^{(0)}(x) + \sum_{j=1}^{\infty} \alpha_j f^{(j)}(x)$$

(3.29)

where $f^{(0)}(x)$ is a particular solution of (3.2), (3.3) and the $\alpha_j$ are arbitrary constants.

**Example 6:**
Consider the integral equation

$$f(x) = \lambda \int_0^\pi \sin(x - y) f(y) dy + g(x).$$

(3.30)

Find

(i) the values of $\lambda$ for which it has a unique solution,

(ii) the solution in this case,

(iii) the resolvent kernel.

For those values of $\lambda$ for which the solution is not unique, find

(iv) a condition which $g(x)$ must satisfy in order for a solution to exist,

(v) the general solution in this case.

The solution proceeds as follows. Expand the kernel:

$$f(x) = \lambda \int_0^\pi \sin(x - y) f(y) dy + g(x)$$

$$= \lambda \int_0^\pi (\sin x \cos y - \cos x \sin y) f(y) dy + g(x)$$

and hence it is clear that

$$K(x, y) = \sin x \cos y - \cos x \sin y$$

is separable. Thus

$$f(x) = \lambda \left[ \sin x \int_0^\pi f(y) \cos y dy - \cos x \int_0^\pi f(y) \sin y dy \right] + g(x),$$

and so write

$$c_1 = \int_0^\pi f(y) \cos y dy,$$

(3.31)

$$c_2 = \int_0^\pi f(y) \sin y dy,$$

(3.32)

which gives

$$f(x) = \lambda [c_1 \sin x - c_2 \cos x] + g(x).$$

(3.33)

Substituting this value of $f(x)$ into (3.31) gives

$$c_1 = \int_0^\pi \left\{ \lambda [c_1 \sin y - c_2 \cos y] + g(y) \right\} \cos y dy.$$
\[ = \lambda c_1 \int_0^\pi \sin y \cos y \, dy - \lambda c_2 \int_0^\pi \cos^2 y \, dy + \int_0^\pi g(y) \cos y \, dy. \]

Defining
\[ g_1 = \int_0^\pi g(y) \cos y \, dy, \]
and noting the values of the integrals
\[ \int_0^\pi \sin y \cos y \, dy = \frac{1}{2} \int_0^\pi \sin 2y \, dy = \left[ -\frac{1}{4} \cos 2y \right]_0^\pi = 0, \]
\[ \int_0^\pi \cos^2 y \, dy = \frac{1}{2} \int_0^\pi (1 + \cos 2y) \, dy = \frac{1}{2} \left[ y + \frac{1}{2} \sin 2y \right]_0^\pi = \frac{1}{2} \pi, \]
yields
\[ c_1 = -\frac{1}{2} \pi \lambda c_2 + g_1. \]

Repeating this procedure, putting \( f(x) \) from (3.33) into (3.32), gives
\[ c_2 = \int_0^\pi \{ \lambda [c_1 \sin y - c_2 \cos y] + g(y) \} \sin y \, dy \]
\[ = \lambda c_1 \int_0^\pi \sin^2 y \, dy - \lambda c_2 \int_0^\pi \cos y \sin y \, dy + \int_0^\pi g(y) \sin y \, dy. \]
Observing that
\[ \int_0^\pi \sin^2 y \, dy = \frac{1}{2} \int_0^\pi (1 - \cos 2y) \, dy = \frac{1}{2} \left[ y - \frac{1}{2} \sin 2y \right]_0^\pi = \frac{1}{2} \pi, \]
and writing
\[ g_2 = \int_0^\pi g(y) \sin y \, dy, \]
then we obtain
\[ c_2 = \frac{1}{2} \pi \lambda c_1 + g_2. \]

Thus, there is a pair of simultaneous equations for \( c_1, c_2 \):
\[ c_1 + \frac{1}{2} \pi \lambda c_2 = g_1, \quad -\frac{1}{2} \pi \lambda c_1 + c_2 = g_2, \]
or in matrix notation
\[ \begin{bmatrix} 1 & \frac{1}{2} \pi \lambda \\ -\frac{1}{2} \pi \lambda & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{3.34} \]

**Case (i):** These equations have a unique solution provided
\[ \det \begin{bmatrix} 1 & \frac{1}{2} \pi \lambda \\ -\frac{1}{2} \pi \lambda & 1 \end{bmatrix} \neq 0, \]
i.e.
\[ 1 + \frac{1}{4} \pi^2 \lambda^2 \neq 0 \quad \text{or} \quad \lambda \neq \pm \frac{2i}{\pi}. \]
In this case
\[
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} =
\begin{bmatrix}
  1 & \frac{1}{2} \pi \lambda \\
  -\frac{1}{2} \pi \lambda & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
  g_1 \\
  g_2
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \frac{1}{1 + \frac{1}{4} \pi^2 \lambda^2}
\begin{bmatrix}
  1 & -\frac{1}{2} \pi \lambda \\
  \frac{1}{2} \pi \lambda & 1
\end{bmatrix}
\begin{bmatrix}
  g_1 \\
  g_2
\end{bmatrix} = \frac{1}{1 + \frac{1}{4} \pi^2 \lambda^2}
\begin{bmatrix}
  g_1 - \frac{1}{2} \pi \lambda g_2 \\
  \frac{1}{2} \pi \lambda g_1 + g_2
\end{bmatrix}
\]
\[
= \frac{1}{1 + \frac{1}{4} \pi^2 \lambda^2} \int_0^{\pi} \left[ \cos y - \frac{1}{2} \pi \lambda \sin y \right] g(y) dy.
\]

Hence,
\[
f(x) = \lambda [c_1 \sin x - c_2 \cos x] + g(x)
\]
\[
= \frac{\lambda}{1 + \frac{1}{4} \pi^2 \lambda^2}
\times \left[ \int_0^{\pi} \left[ \sin x \left( \cos y - \frac{1}{2} \pi \lambda \sin y \right) - \cos x \left( \frac{1}{2} \pi \lambda \cos y + \sin y \right) \right] g(y) dy + g(x)
\]
or
\[
f(x) = \frac{\lambda}{1 + \frac{1}{4} \pi^2 \lambda^2} \int_0^{\pi} \left[ \sin (x - y) - \frac{1}{2} \pi \lambda \cos (x - y) \right] g(y) dy + g(x).
\]
This is the required solution, and we can observe that the resolvent kernel is
\[
R(x, y, \lambda) = \frac{\sin (x - y) - \frac{1}{2} \pi \lambda \cos (x - y)}{1 + \frac{1}{4} \pi^2 \lambda^2}.
\]

Case (ii): If
\[
\det \begin{bmatrix}
  1 & \frac{1}{2} \pi \lambda \\
  -\frac{1}{2} \pi \lambda & 1
\end{bmatrix} = 0
\]
i.e.
\[
\lambda = \pm \frac{2i}{\pi}
\]
then there is either no solution or infinitely many solutions. With this, (3.34) becomes
\[
\begin{bmatrix}
  1 & \pm i \\
  \mp i & 1
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} =
\begin{bmatrix}
  g_1 \\
  g_2
\end{bmatrix}.
\]
Solving these equations using row operations, $R2 \leftrightarrow R2 \pm iR1$, gives
\[
\begin{bmatrix}
  1 & \pm i \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} =
\begin{bmatrix}
  g_1 \\
  g_2 \pm ig_1
\end{bmatrix}
\]
or
\[
c_1 \pm ic_2 = g_1,
\]
\[
0 = g_2 \pm ig_1.
\]
The second equation places a restriction on \( g(x) \), which by definition of the \( g_i \), is

\[
\int_0^\pi (\sin y \pm i \cos y) g(y) \, dy = 0. \tag{3.35}
\]

This is the condition that \( g(x) \) must satisfy for the integral equation to be soluble, i.e. if \( g(x) \) does not satisfy this, then (3.30) does not have a solution. Suppose this condition holds then we can set \( c_2 \) to take any arbitrary constant value, \( c_2 = \alpha \), say. Thus,

\[ c_1 = \mp i \alpha + g_1, \]

and hence from (3.33), the solution of (3.30) is, when \( \lambda = \pm 2i/\pi \),

\[
f(x) = \pm \frac{2i}{\pi} [(\mp i \alpha + g_1) \sin x - \alpha \cos x] + g(x)
\]

\[
= \frac{2\alpha}{\pi} (\sin x \mp i \cos x) + \frac{2i}{\pi} g_1 \sin x + g(x)
\]

for arbitrary \( \alpha \), with constraint \( g_2 = \mp ig_1 \) or equivalently (3.35).
We arrived at the above conclusions via simple row operations. Fredholm’s theorem would have also told us the same information regarding constraints on \( g_1 \) and \( g_2 \).