[UNDER CONSTRUCTION]

§3 Connection on a vector bundle

3.1 Definition of a connection

Consider a vector bundle $E \to B$.

Denote by $\mathfrak{X}(M)$ the space of all vector fields on M.

Definition 3.1. A *connection* in a vector bundle $E \to B$ is defined as a map (referred to as the *covariant derivative*)

$$\mathfrak{X}(U) \times \Gamma(U, E) \to \Gamma(U, E)$$

for each open $U \subset B$, notation: $(X, \boldsymbol{u}) \mapsto \nabla_X \boldsymbol{u}$, satisfying the following properties:

- $\nabla_{X+Y} \boldsymbol{u} = \nabla_X \boldsymbol{u} + \nabla_Y \boldsymbol{u}, \quad \nabla_{fX} \boldsymbol{u} = f \, \nabla_X \boldsymbol{u}$
  (linearity over functions);

- $\nabla_X (\boldsymbol{u} + \boldsymbol{w}) = \nabla_X \boldsymbol{u} + \nabla_X \boldsymbol{w}, \quad \nabla_X (c\boldsymbol{u}) = c \, \nabla_X \boldsymbol{u}$
  (linearity over numbers);

- $\nabla_X (f\boldsymbol{u}) = (\partial_X f)\boldsymbol{u} + f \, \nabla_X \boldsymbol{u}$
  (Leibniz rule)

and compatible with restrictions.

The last property means that if $U \subset W$ and $X \in \mathfrak{X}(W)$, $\boldsymbol{u} \in \Gamma(W, E)$, then $(\nabla_X \boldsymbol{u})|U = \nabla_{X|U}(\boldsymbol{u}|U)$. It will allow us to describe connections using local data.

Example 3.1. For $E = T\mathbb{R}^n$ consider the "ordinary" derivative $\partial_X Y = X^i (\partial_i Y^j) \boldsymbol{e}_j$ in Cartesian coordinates, where $X, Y \in \mathfrak{X}(\mathbb{R}^n)$. It satisfies the above properties (check!). Hence ∂ may be treated as ∇, a covariant derivative in $T\mathbb{R}^n$ (a covariant derivative of vector fields on a Euclidean space). Notice that it has a simple appearance in affine coordinates only. It assumes a more complicated form as soon as we pass to curvilinear coordinates, see examples in Section 1.

Example 3.2. Consider an arbitrary trivial bundle $E \cong B \times \mathbb{R}^k$ and choose a basis of sections $\boldsymbol{e}_i \in \Gamma(B, E)$, $i = 1, \ldots, k$. Then it is possible to set $\nabla_X \boldsymbol{e}_i := 0$ for all i and arbitrary vector fields X on B. This defines a connection on the trivial bundle $E \to B$, which is called the *trivial connection.* For an arbitrary section $\boldsymbol{u} = \boldsymbol{e}_i u^i$, the formula is $\nabla_X \boldsymbol{u} = \boldsymbol{e}_i(\partial_X u^i)$.

Remark 3.1. Trivial connection can be defined for trivial bundles only. On the other hand, for a trivial bundle it is possible to define a connection that is not trivial. The last claim requires some elaboration. It means that for a given global basis of sections the covariant derivative of basis sections is not zero, and furthermore, by passing to a different basis it is not possible to "kill" all the covariant derivatives of basis section (and thus make it look as trivial). We shall analyze this later.

Example 3.3. Consider the tautological bundle $E \to \mathbb{R}P^n$ over the projective space $\mathbb{R}P^n$. Each fiber is just a line in $\mathbb{R}^{n+1}$. More precisely, for a point $\boldsymbol{z} = (z^{:} \ldots : z^{n+1}) \in \mathbb{R}P^n$ (where we use homogeneous coordinates) the fiber $E_{\boldsymbol{z}}$ consists of all vectors belonging to the equivalence class $\boldsymbol{z}$. A section of E is just a vector-function on $\mathbb{R}P^n$ (taking values in $\mathbb{R}^{n+1}$) such that for each point $\boldsymbol{z} \in \mathbb{R}P^n$ the vector $\boldsymbol{u}(\boldsymbol{z})$ belongs to $\boldsymbol{z}$. One can define a covariant derivative on E as follows: for a section $\boldsymbol{u}$ considered as a vector-function on $\mathbb{R}P^n$ take the ordinary derivative $\partial_X \boldsymbol{u}$, which is a vector-function not necessarily representing a section of E (i.e., the vector $\partial_X \boldsymbol{u}$ at a given point $\boldsymbol{z}$ does not necessarily belong to $\boldsymbol{z}$), and apply the orthogonal projection $P = P_{\boldsymbol{z}}$ onto the subspace $\boldsymbol{z} \subset \mathbb{R}^{n+1}$ for each $\boldsymbol{z} \in \mathbb{R}P^n$. By definition, $\nabla_X \boldsymbol{u} := P(\partial_X \boldsymbol{u})$. One can check that all axioms are satisfied.

Consider a particular connection on a vector bundle E. Since the covariant derivative $\nabla_X \boldsymbol{u}$ is linear over functions w.r.t. a vector field X, it defines a 1-form with values in E (as the map $X \mapsto \nabla_X \boldsymbol{u}$). This 1-form is called the *covariant differential* of a section $\boldsymbol{u}$ and denoted $\nabla \boldsymbol{u}$. Thus, the covariant differential is defined by the relation $\langle \nabla \boldsymbol{u}, X \rangle = \nabla_X \boldsymbol{u}$ for any vector field X. The properties of the covariant derivatives $\nabla_X \boldsymbol{u}$ are equivalent to the following properties of the covariant differential:

- $\nabla(\boldsymbol{u} + \boldsymbol{w}) = \nabla \boldsymbol{u} + \nabla \boldsymbol{w}, \quad \nabla(c\boldsymbol{u}) = c\,\nabla \boldsymbol{u}$
  (linearity over numbers);

- $\nabla(f\boldsymbol{u}) = df\,\boldsymbol{u} + f\,\nabla \boldsymbol{u}$
  (Leibniz rule)

(in the first term we have the product of an ordinary 1-form and a section of E, which gives an E-valued 1-form).

3.2 Local connection 1-form and the existence theorem

Consider an open domain $U \subset B$ such that a local basis of sections (otherwise called a local frame) of E over U exists. Denote it $\boldsymbol{e}_i$, where $i = 1, \dots, k$. Every section $\boldsymbol{u} \in \Gamma(U, E)$ can be expanded as

$$\boldsymbol{u} = \boldsymbol{e}_i u^i$$

where the components u^i are just smooth functions on U. For a given connection ∇ on E we have

$$\nabla_X \boldsymbol{u} = \nabla_X(\boldsymbol{e}_i u^i) = \nabla_X \boldsymbol{e}_i \, u^i + \boldsymbol{e}_i \, \partial_X(u^i)$$

(where we used the Leibniz rule). Expanding $\nabla_X \boldsymbol{e}_i$ over the given frame we arrive at

$$\nabla_X \boldsymbol{e}_i = \boldsymbol{e}_j \widetilde{A^j}_i$$

where the coefficients $\widetilde{A^j}_i$ depend linearly on a vector field X, i.e.,

$$\widetilde{A^j}_i = A^j{}_i(X) \quad \text{and} \quad A^j{}_i(fX) = f \, A^j{}_i(X)$$

for any $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$. Therefore we have a collection of 1-forms $A^j{}_i$, which we shall write as a matrix $A = (A^j{}_i)$. This is the same thing as a 1-form taking values in matrices. Using the notion of the covariant differential we may write

$$\nabla \boldsymbol{e}_i = \boldsymbol{e}_j A^j{}_i \, .$$

Definition 3.2. The matric-valued 1-form A is called the *(local) connection* 1-*form* w.r.t. a local frame $(\boldsymbol{e}_i)$.

Assuming for convenience that in the open domain U we can define local coordinates x^a we may write the local connection 1-form A as

$$A = A_a dx^a$$

where the components A_a are matrices, $A_a = (A^j_{ai})$. The definition of local connection forms can be compactly written in the matrix notation:

$$\nabla \boldsymbol{e} = \boldsymbol{e} A$$

where $\boldsymbol{e} = (\boldsymbol{e}_i)$ is a row-vector of sections; at the LHS, $\nabla \boldsymbol{e}$ means the operation ∇ applied to each element of the row-vector, and at the RHS, $\boldsymbol{e}A$ is the product of a row-vector and a matrix.

Theorem 3.1. *Suppose A_α is the connection 1-form w.r.t. a local frame $\boldsymbol{e}_{(\alpha)}$ and A_β is the connection 1-form w.r.t. a local frame $\boldsymbol{e}_{(\beta)}$, where $\boldsymbol{e}_{(\beta)} = \boldsymbol{e}_{(\alpha)} g_{\alpha\beta}$. Then*

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta} g_{\alpha\beta}^{-1}\,.$$

Proof. Using the matrix notation we have $\nabla \boldsymbol{e}_{(\alpha)} = \boldsymbol{e}_{(\alpha)} A_\alpha$ and $\nabla \boldsymbol{e}_{(\beta)} = \boldsymbol{e}_{(\beta)} A_\beta$. On the other hand,

$$\begin{aligned}\nabla \boldsymbol{e}_{(\beta)} = \nabla(\boldsymbol{e}_{(\alpha)} g_{\alpha\beta}) = (\nabla \boldsymbol{e}_{(\alpha)}) g_{\alpha\beta} + \boldsymbol{e}_{(\alpha)} dg_{\alpha\beta} = \boldsymbol{e}_{(\alpha)} A_\alpha g_{\alpha\beta} + \boldsymbol{e}_{(\alpha)} dg_{\alpha\beta} = \\ \boldsymbol{e}_{(\beta)} g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + \boldsymbol{e}_{(\beta)} g_{\alpha\beta}^{-1} dg_{\alpha\beta} = \boldsymbol{e}_{(\beta)} (g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta})\,.\end{aligned}$$

Therefore

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

or

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta} g_{\alpha\beta}^{-1}\,.$$

□

Recall that an *affine space* X is defined as a set, whose elements are called points, endowed with an operation of 'difference' $X \times X \to V$ where V is a vector space V called the associated vector space of the affine space X. The difference of points $P, Q \in X$ is denoted $P - Q$ or $\overrightarrow{QP}$ and should satisfy the following properties: for any three points P, Q, and R,

$$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = 0$$

(in particular, it follows that $\overrightarrow{PP} = 0$ for any P and $\overrightarrow{PQ} = -\overrightarrow{QP}$ for any P and Q); for any point P and any vector $\boldsymbol{u} \in V$ there exists a unique point Q such that $\boldsymbol{u} = \overrightarrow{PQ}$. Any vector space can be regarded as an affine space w.r.t. the operation $V \times V \to V$, $(\boldsymbol{a}, \boldsymbol{b}) \mapsto \boldsymbol{a} - \boldsymbol{b}$. A subset of a vector space makes an affine space if the differences of all its elements is a linear subspace. These are typical situations where affine spaces arise.

Theorem 3.2. *Connections on a given vector bundle $E \to B$ make an affine space with the associated vector space $\Omega^1(B, \operatorname{End} E)$, the space of 1-forms with values in the linear operators on the fibers of E.*

Proof. Two approaches are possible. Consider two connections, ∇ and ∇'. We can either consider their difference as an operation $\nabla - \nabla'$ acting on sections of E and giving a 1-form with values in E, or consider the local connection 1-forms specifying ∇ and ∇' and consider the difference of these local 1-forms.

In the first approach (which is more abstract) we use the Leibniz rule satisfied by both ∇ and ∇':

$$\nabla(f\boldsymbol{u}) = df\,\boldsymbol{u} + f\nabla\boldsymbol{u}$$
$$\nabla'(f\boldsymbol{u}) = df\,\boldsymbol{u} + f\nabla'\boldsymbol{u}$$

Taking the difference gives

$$(\nabla - \nabla')(f\boldsymbol{u}) = f(\nabla - \nabla')(\boldsymbol{u})\,,$$

i.e., the linearity over functions. Therefore for each fiber $E_{\boldsymbol{x}}$ of $E \to B$ the operation $\nabla - \nabla'$ defines a linear transformation from $E_{\boldsymbol{x}}$ to the space $E_{\boldsymbol{x}}$ 'multiplied by 1-forms' (i.e., if a tangent vector is given, it can be used to fill in the slot obtaining just a linear transformation $E_{\boldsymbol{x}} \to E_{\boldsymbol{x}}$).

The second approach allowed to see the details better. Consider an open cover $\mathfrak{U} = (U_\alpha)$ with local frames $\boldsymbol{e}_\alpha$ on open sets U_α. Let A_α and A'_α denote local connection 1-forms for ∇ and ∇' respectively. They satisfy the same transformation law:

$$A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1}\,,$$
$$A'_\alpha = g_{\alpha\beta}A'_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1}\,.$$

By taking the difference we arrive at

$$(A_\alpha - A'_\alpha) = g_{\alpha\beta}(A_\beta - A'_\beta)g_{\alpha\beta}^{-1}\,,$$

since the inhomogeneous terms cancel. This is exactly the transformation law for the matrix of a linear operator on fibers of E. Therefore the collection of local matrix-valued 1-forms $(A_\alpha - A'_\alpha)$ defines a (global) 1-form on B with values in the linear operators on the fibers of E. (The relation with the first approach is that the matrix-valued 1-form $A_\alpha - A'_\alpha$ is the matrix of the linear operator $\nabla - \nabla'$ w.r.t. the frame $\boldsymbol{e}_\alpha$.)

In both approaches we see that the difference of connections $\nabla - \nabla'$ is naturally defined as a 1-form with values in fiberwise linear transformations $E \to E$ and all axioms in the definition of an affine space are satisfied. $\square$

Theorem 3.3. *For any vector bundle $E \to M$, real or complex, there exists a connection.*

Proof. We need to construct a collection of matrix-valued 1-forms A_α such that

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta} g_{\alpha\beta}^{-1} \,.$$

Consider the 1-forms $-dg_{\alpha\beta} g_{\alpha\beta}^{-1}$ defined on the intersections $U_{\alpha\beta}$ and set

$$A_\alpha := \sum_\beta f_\beta(-dg_{\alpha\beta} g_{\alpha\beta}^{-1})$$

where (f_α) is a partition of unity subordinate to the open cover (U_α). Each term in the sum is originally defined on $U_{\alpha\beta}$, but since f_β is defined everywhere and vanishes outside of U_β, each term extends to the whole U_α by zero and therefore the sum is defined on U_α. We shall show that this collection of 1-forms is as desired. Consider the cocycle identity for the transition functions:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \,.$$

By differentiating we obtain

$$dg_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} dg_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma} dg_{\gamma\alpha} = 0 \,.$$

We can use the cocycle identity once again to rewrite this as

$$dg_{\alpha\beta} g_{\beta\alpha} + g_{\alpha\beta} dg_{\beta\gamma} g_{\beta\gamma}^{-1} g_{\beta\gamma} g_{\gamma\alpha} + g_{\alpha\beta} g_{\beta\gamma} dg_{\gamma\alpha} g_{\gamma\alpha}^{-1} g_{\gamma\alpha} = 0$$

(where we also inserted the identities $g_{\beta\gamma}^{-1} g_{\beta\gamma} = 1$ and $g_{\gamma\alpha}^{-1} g_{\gamma\alpha} = 1$) and then as

$$dg_{\alpha\beta} g_{\beta\alpha} + g_{\alpha\beta}(dg_{\beta\gamma} g_{\beta\gamma}^{-1}) g_{\beta\alpha} + g_{\alpha\gamma}(dg_{\gamma\alpha} g_{\gamma\alpha}^{-1}) g_{\gamma\alpha} = 0$$

or

$$dg_{\alpha\beta} g_{\beta\alpha} + g_{\alpha\beta}(dg_{\beta\gamma} g_{\beta\gamma}^{-1}) g_{\alpha\beta}^{-1} + g_{\alpha\gamma}(dg_{\gamma\alpha} g_{\gamma\alpha}^{-1}) g_{\alpha\gamma}^{-1} = 0$$

.... By multiplying from the right by $g_{\gamma\alpha}^{-1}$ we obtain

$$dg_{\alpha\beta} g_{\beta\gamma} + g_{\alpha\beta} dg_{\beta\gamma} + g_{\alpha\beta} g_{\beta\gamma} dg_{\gamma\alpha} g_{\gamma\alpha}^{-1} = 0 \,.$$

We can use the cocycle identity once again to rewrite this formula first as

$$dg_{\alpha\beta} g_{\alpha\beta}^{-1} g_{\alpha\beta} g_{\beta\gamma} + g_{\alpha\beta} dg_{\beta\gamma} g_{\beta\gamma} g_{\beta\gamma}^{-1} + g_{\alpha\gamma} dg_{\gamma\alpha} g_{\gamma\alpha}^{-1} = 0 \,.$$

(we also inserted the identities $g_{\alpha\beta}^{-1} g_{\alpha\beta} = 1$ and $g_{\beta\gamma} g_{\beta\gamma}^{-1} = 1$) and then as

□

Remark: Statement: every vector bundle is a subbundle of a trivial vector bundle.

For subbundles of a trivial bundle.

Suppose there is a morphism of fiber bundles $E \to B \times \mathbb{R}^N$..........

Theorem 3.4. *For a subbundle of a trivial bundle, the construction* $P(\partial_X u)$ *defines a connection.*

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