Introduction

It is often the case in EIT that we would like to locate a foreign object or inclusion within a domain, in which the conductivity differs from the known background conductivity. At the boundary of the inclusion there is a jump change in the conductivity distribution and the factorization method is used to locate this jump change. The idea for the method was first introduced by Colton and Kirsch [2] to locate the boundary of a 2-D object by solving a far-field scattering problem. A different approach was used by Kirsch [5] to solve the same problem in 3-D, in which a factorization of the far-field operator was utilized, hence the name of the method. The method was first applied to the problem of EIT by Brühl [1] in which a factorization of the Neumann to Dirichlet operator was found.

Method

Let \( \Omega \subset \mathbb{R}^3 \) be our domain with homogeneous background conductivity and let \( D \subset \Omega \) be an inclusion with different conductivity to that of the homogeneous background. For the case of homogeneous conductivity throughout \( \Omega \), define the reference Neumann to Dirichlet operator as \( \Lambda_1 \) and let \( \Lambda_\sigma \) be the Neumann to Dirichlet operator when an inclusion is present. It has been shown in [1] that for a point \( z \in \Omega \)

\[
  z \in D \iff g_{z,d} \in \mathcal{R} \left( |\Lambda_\sigma - \Lambda_1|^\frac{1}{2} \right),
\]

where \( g_{z,d}(x) \) is the dipole potential at point \( z \) with dipole moment \( d \in \mathbb{R}^3 \) and measured with respect to a point \( x \in \partial \Omega \). Since \( \Lambda_\sigma - \Lambda_1 : L^2(\partial \Omega) \rightarrow L^2(\partial \Omega) \) is compact we can use the Picard criterion to test if \( g_{z,d} \) is in the required range, i.e.

\[
  g_{z,d} \in \mathcal{R} \left( |\Lambda_\sigma - \Lambda_1|^\frac{1}{2} \right) \iff \sum_{k=1}^{\infty} \left( \frac{g_{z,d} \cdot v_k}{\lambda_k} \right)^2 < \infty,
\]

where \( v_k \) are the eigenfunctions and \( \lambda_k \) the eigenvalues of \( |\Lambda_\sigma - \Lambda_1|^\frac{1}{2} \). The above theory has been proved for the theoretical continuous case but in practice we will only ever have a discrete set of current stimulations and voltage measurements, therefore we must find a discrete approximation to the operator \( \Lambda_\sigma - \Lambda_1 \) and compute its SVD to approximate the eigenvalues and eigenfunctions of \( |\Lambda_\sigma - \Lambda_1|^\frac{1}{2} \). We then define a grid of test points \( z \in \Omega \) for which we compute the analytic (if known) or numerical dipole potential. To test the convergence of the Picard series, linear regression parameters are used to find a relation for \( \lambda_k \) and \( \langle g_{z,d}, v_k \rangle^2 \) as outlined in [4].

We now look at the method applied to a semi-infinite half-space with a square planar electrode array on the top surface. This type of array is common in geophysics and has been implemented in [6] with applications in breast cancer detection. For the 3-D half-space domain used in the following simulated results, the analytic dipole potential is known to be

\[
  g_{x,d}(x) = \frac{(x - z) \cdot d}{|x - z|^3},
\]

where the points \( x \) are taken to be the centre points of the electrodes and \( d = (0,0,-1)^T \). To simulate a half-space, a large domain was used so that the far-field potentials were negligible but in order to get a refined mesh at the points of interest the same mesh had to be used for the homogeneous and inhomogeneous cases. To overcome this "inverse crime", infinite elements should be implemented (see [7]) to ensure that far-field boundary conditions are defined on the truncated domain boundaries, resulting in a finer mesh for areas of interest.

Results

Figure 1: Log plot of the singular values of \( A \approx \Lambda_\sigma - \Lambda_1 \) (left) and the corresponding reconstruction (right) for a spherical inclusion with constant conductivity \( \sigma = 1.1 \) centred at \((-1,-1,-1)\) with radius 0.75 using an \( 8 \times 8 \) grid of electrodes.

The figure shows that the singular values decay in clusters and after a particular tolerance value they decay more gradually and in larger clusters. It has been found that discarding singular values whose value are less than this tolerance gives much better reconstruction of the inclusion. Regarding the reconstruction, it can be observed that the location and the shape of the inclusion is well approximated.

Conclusions and Further Work

The factorization method is a fairly simple method to implement (particularly when the dipole potential for the domain is known) and from the preliminary results presented it can be seen that it gives a good indication of the location of an inclusion of differing conductivity.

In order to make this method practical, the following areas still need further investigation:

- Extension to general domain shapes by numerical calculation of dipole potentials using boundary element method or following the work in [3]. An alternative approach may be the use of conformal mappings;
- Optimization of electrode arrays and stimulation/measurement patterns for particular applications;
- Optimization of eigenvalue truncation tolerance for given domains and electrode patterns;
- Application of factorization method as a priori information for other methods.

References