MATH31001/MATH41001/MATH61001: LINEAR ANALYSIS

Chapter 4: Linear Operators and Spectra

Basic Ideas

Definition. Let V and V' be normed vector spaces. A *linear operator* is a map $T : V \to V'$ such that
$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$
for all $x, y \in V$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

The following is analogous to Proposition 3.1 for linear functionals. To make notation less cumbersome, we shall denote both the norm on V and the norm on V' by $\|\cdot\|$.

We shall be interested in linear operators $T : V \to V'$ which are continuous, i.e., those for which $\lim_{n\to+\infty} \|x_n - x\| = 0$ implies $\lim_{n\to+\infty} \|T(x_n) - T(x)\| = 0$.

Definition. We say that $T : V \to V'$ is bounded if there exists $M \geq 0$ such that
$$\|T(x)\| \leq M\|x\|, \quad \text{for all } x \in V.$$

Theorem 4.1. *Let $T : V \to V'$ be a linear operator between the normed vector spaces V and V'. Then T is continuous if and only if T is bounded.*

Proof. Exercise. (It is essentially the same as the proof of Proposition 3.1 for linear functionals.) □

Definition. If $T : V \to V'$ is a bounded linear operator then we define its norm $\|T\|$ by
$$\|T\| = \sup_{\|x\|=1} |T(x)|.$$

By Theorem 4.1, this is finite and an equivalent definition is
$$\|T\| = \sup_{\|x\|\neq 0} \frac{\|T(x)\|}{\|x\|}.$$

(N.B. it still has to be proved that this is a norm.)

An immediate consequence of the definition is the following estimate (cf. Corollary 3.1.2).

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Corollary 4.1.2. *For all $x \in V$,*

$$\|T(x)\| \leq \|T\| \; \|x\|.$$

Definition. We define $B(V, V')$ to be the set of all bounded linear operators $T : V \to V'$. (If V is over $\mathbb{R}, \mathbb{C}$ and $V' = \mathbb{R}, \mathbb{C}$ then $B(V, V') = V^*$.)

Proposition 4.2. *If V' is a Banach space then so is $B(V, V')$. (We do not need to assume that V is a Banach space.*

Proof. Exercise. (It is essentially the same as the proof of Proposition 3.2 for linear functionals.) □

Now we will consider the case where $V = V'$. We then use the simpler notation

$$B(V) = B(V, V).$$

If $T, T' \in B(V)$ we write TT' for their composition: $(TT')(x) = T(T'(x))$. This is also a linear operator.

Proposition 4.3. *If $T, T' \in B(V)$ then $\|TT'\| \leq \|T\| \; \|T'\|$. (In particular, $TT' \in B(V)$.)*

Proof. Applying Corollary 4.1.2 twice, we have

$$\|TT'(x)\| = \|T(T'(x))\| \leq \|T\| \; \|T'(x)\| \leq \|T\| \; \|T'\| \; \|x\|.$$

Thus TT' is bounded (so $TT' \in B(V, V)$) and $\|TT'\| \leq \|T\| \; \|T'\|$. □

Example. If $V = \mathbb{R}^n$ then $T \in B(V)$ is given by an $n \times n$ real matrix.
Similarly, If $V = \mathbb{C}^n$ then $T \in B(V)$ is given by an $n \times n$ complex matrix.

Example. Let $V = l^p$, $1 \leq p < \infty$. A particular $T \in B(V)$ is the *shift* $T : l^p \to l^p$ defined by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

Then ($x = (x_1, x_2, x_3, \dots)$)

$$\|T(x)\| = \left(\sum_{i=2}^{\infty} |x_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} = \|x\|,$$

so $\|T\| \leq 1$.

Now put $x = (0, x_2, x_3, \dots)$, so $T(x) = (x_2, x_3, x_4, \dots)$. For this x, it is clear that $\|T(x)\| = \|x\|$, so we have

$$\|T\|\|x\| \geq \|T(x)\| = \|x\|,$$

giving $\|T\| \geq 1$. Therefor $\|T\| = 1$.

Definition. We say that $T \in B(V, V')$ is an an isometry if $\|T(x)\| = \|x\|$, for all $x \in V$.

We have now got back to a definition we made in Chapter 3: $T : V \to V'$ is an isometric isomorphism if T is a linear isometry which is also a bijection.

Let H be a Hilbert space (with inner product $\langle \cdot, \cdot \rangle$) and let $T : H \to H$ be a bounded linear operator. We want to define a bounded linear operator $T^* : H \to H$, called the *adjoint* of T, such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \text{for all } x, y \in H.$$

As we shall see, this generalizes the notion of transpose for real matrices or Hermitian transpose for complex matrices.

Let us see that we can do this. Fix $y \in H$ and consider the linear map

$$H \to \mathbb{C} : x \mapsto \langle Tx, y \rangle.$$

Since, using the Cauchy-Schwarz inequality,

$$|\langle Tx, y \rangle| \leq \|Tx\| \|y\| \leq \|T\| \|x\| \|y\|,$$

this linear map is a bounded linear functional. Thus, by the Riesz Representation Theorem, there is a unique $y' \in H$ such that

$$\langle Tx, y \rangle = \langle x, y' \rangle$$

and we shall define $T^* : H \to H$ by

$$T^*(y) = y'.$$

Before we show that T^* is a bounded linear operator, let's give some examples.

Example 1. Let $H = \mathbb{R}^n$. A linear operator in $B(\mathbb{R}^n)$ is given by an $n \times n$ real matrix $A = (a_{ij})$ with $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$. Now

$$\langle Ax, y \rangle = \sum_{i=1}^n (Ax)_i y_i = \sum_{i,j=1}^n a_{ij} x_j y_i = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} y_i \right) x_j = \sum_{j=1}^n (A^t y)_j x_j = \langle x, A^t y \rangle.$$

Thus the adjoint of A is A^t (the transpose of A).

Example 2. Let $H = \mathbb{C}^n$. A linear operator in $B(\mathbb{C}^n)$ is given by an $n \times n$ complex matrix $A = (a_{ij})$ with $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$. Now

$$\begin{aligned}\langle Ax, y \rangle &= \sum_{i=1}^n (Ax)_i \overline{y_i} = \sum_{i,j=1}^n a_{ij} x_j \overline{y_i} = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} \overline{y_i} \right) x_j \\ &= \sum_{j=1}^n \overline{\left(\sum_{i=1}^n \overline{a_{ij}} y_i \right)} x_j = \sum_{j=1}^n \overline{(A^* y)_j} x_j = \langle x, A^* y \rangle,\end{aligned}$$

where A^* is the Hermitian transpose of A, i.e., the matrix with i,j-entry $\overline{a_{ji}}$. Thus the adjoint of A is the Hermitian transpose of A.

Example 3. Let $H = l^2$. If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ then

$$\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots .$$

Now let T be the shift operator we considered above:

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Then

$$\langle Tx, y\rangle = x_2\overline{y_1} + x_3\overline{y_2} + \cdots = (x_1 \times 0) + x_2\overline{y_1} + x_3\overline{y_2} + \cdots .$$

Equating this to $\langle x, T^*y\rangle$ gives

$$T^*y = (0, y_1, y_2, \dots).$$

Before we show that the adjoint T^* is indeed a bounded linear operator, we need a lemma.

Lemma 4.4. $T^{**} = T$.

Proof. Of course, T^{**} means $(T^*)^*$, i.e., the adjoint of the adjoint. By its definition,

$$\langle T^*y, x\rangle = \langle y, T^{**}x\rangle, \quad \text{for all } x, y \in H.$$

But

$$\langle T^*y, x\rangle = \overline{\langle x, T^*y\rangle} = \overline{\langle Tx, y\rangle} = \langle y, Tx\rangle.$$

Subtracting one equation from the other gives

$$0 = \langle y, Tx - T^{**}x\rangle, \quad \text{for all } x, y \in H.$$

Substituting $y = Tx - T^{**}x$ gives

$$\|Tx - T^{**}x\|^2 = \langle Tx - T^{**}x, Tx - T^{**}x\rangle = 0.$$

Thus $Tx = T^{**}x$, for all $x \in H$, i.e., $T = T^{**}$. □

Proposition 4.5. *$T^* : H \to H$ is a bounded linear operator with $\|T^*\| = \|T\|$.*

Proof.
Claim. $T^* : H \to H$ is linear: For all $x, y_1, y_2 \in H$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$),

$$\begin{aligned}\langle x, T^*(\lambda y_1 + \mu y_2)\rangle &= \langle Tx, \lambda y_1 + \mu y_2\rangle \\ &= \overline{\lambda}\langle Tx, y_1\rangle + \overline{\mu}\langle Tx, y_2\rangle \\ &= \overline{\lambda}\langle x, T^*(y_1)\rangle + \overline{\mu}\langle x, T^*(y_2)\rangle \\ &= \langle x, \lambda T^*(y_1) + \mu T^*(y_2)\rangle.\end{aligned}$$

Thus $T^*(\lambda y_1 + \mu y_2) = \lambda T^*(y_1) + \mu T^*(y_2)$.
Claim. $\|T^*\| = \|T\|$*:* Since $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in H$, by setting $x = T^*y$, we have $\langle TT^*y, y\rangle = \langle T^*y, T^*y\rangle$. Thus

$$\begin{aligned}\|T^*y\|^2 &= \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \\ &\leq \|TT^*y\| \, \|y\| \text{ (by the Cauchy-Schwarz inequality)} \\ &\leq \|T\| \, \|T^*y\| \, \|y\|.\end{aligned}$$

In particular, $\|T^*y\| \leq \|T\| \, \|y\|$, for all $y \in H$, i.e., $\|T^*\| \leq \|T\|$. This shows that T^* is bounded.

Since $T^{**} = T$, we see that we also have $\|T\| = \|(T^*)^*\| \leq \|T^*\|$. Thus, $\|T^*\| = \|T\|$, as required. □

Proposition 4.6.

(i) $I^* = I$ *(where* $I : H \to H$ *is the identity transformation);*
(ii) $(ST)^* = T^*S^*$*;*
(iii) $\|T^*T\| = \|TT^*\| = \|T\|^2$.

Proof. Property (i) follow directly from the definition. Properties (ii) and (iii) are exercises on Example Sheet 6.

Definition. We called a bounded linear operator $T : H \to H$ *self-adjoint* if $T^* = T$.

Example. From Example 1 above, an $n \times n$ real matrix A acting on $\mathbb{R}^n$ is self-adjoint if and only if $A^t = A$, i.e. A is symmetric. Fom Example 2, an $n \times n$ complex matrix A acting on $\mathbb{C}^n$ is self-adjoint if and only if A is equal to its Hermitian transpose, i.e., $\overline{a_{ji}} = a_{ij}$.

Example. Let $H = l^2(\mathbb{C})$ and let $(a_i)_{i=1}^\infty \in l^\infty(\mathbb{R})$. Define $T : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by

$$T((x_i)_{i=1}^\infty) = (a_i x_i)_{i=1}^\infty.$$

(Check that $T \in B(l^2(\mathbb{C}))$!) Then, for all $(x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \in l^2(\mathbb{C})$,

$$\begin{aligned}\langle T((x_i)_{i=1}^\infty), (y_i)_{i=1}^\infty\rangle &= \sum_{i=1}^\infty (a_i x_i)\overline{y_i} \\ &= \sum_{i=1}^\infty x_i(\overline{a_i y_i}) \\ &= \langle (x_i)_{i=1}^\infty, T((y_i)_{i=0}^\infty)\rangle.\end{aligned}$$

i.e., $T^* = T$, so T is self-adjoint. (Note that it was important for $(a_i)_{i=1}^\infty$ to heave real entries. □

Spectrum of Operators

Let V be a Banach space over $\mathbb{C}$. (Here it is important that the field is $\mathbb{C}$ (algebraically closed) not $\mathbb{R}$.)

We will say that a bounded linear operator $T \in B(V)$ is *invertible* if there exists $S \in B(V)$ such that $TS = ST = I$ (the identity transformation). If T is invertible then we denote this S by T^{-1} and call it the *inverse* of T.

Recall that an $n \times n$ matrix A has eigenvalue $\lambda \in \mathbb{C}$ if any one of the following equivalent statements holds

(1) there exists $v \in \mathbb{C}^n$, $v \neq 0$, such that $Av = \lambda v$; or, equivalently,
(2) $\det(\lambda I - A) = 0$ (the characteristic equation);
(3) $\lambda I - A$ is *not* invertible.

In infinite dimensions, (2) does not make sense in general, while (1) and (3) make sense but are no longer equivalent. If (1) holds (i.e. $Tx = \lambda x$ for some non-zero $x \in V$) then we will still call λ and eigenvalue but we will focus on condition (3).

Definition. We define the *spectrum* of $T \in B(V)$ to be the set of complex numbers

$$\mathrm{spec}(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) : V \to V \text{ is not invertible}\}.$$

Exercise. Show that if λ is an eigenvalue of T then $\lambda \in \mathrm{spec}(T)$.

Solution. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then, by definition, there exists $x \in V$, $x \neq 0$, such that

$$Tx = \lambda x$$

or, equivalently,

$$(\lambda I - T)x = 0.$$

Suppose that $\lambda \notin \mathrm{spec}(T)$, so that $\lambda I - T$ is invertible. Write $S = (\lambda I - T)^{-1}$. Then

$$x = Ix = S(\lambda I - T)x = S0 = 0,$$

a contradiction, since $x \neq 0$. Therefore, $\lambda \in \mathrm{spec}(T)$, as required.

An example below will show that the spectrum can contain numbers which are not eigenvalues.

Proposition 4.7. *Suppose that V is a Banach space over $\mathbb{C}$ and that $T : V \to V$ is a bounded linear operator. Then spec(T) is a compact (i.e. closed and bounded) set in $\mathbb{C}$. Furthermore, spec$(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$.*

Proof.
Claim: spec$(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$. Suppose that $|\lambda| > \|T\|$; we shall show that $(\lambda I - T) : V \to V$ is invertible (so that $\lambda \notin \mathrm{spec}(T)$). For $N \geq 1$, define

$$S_N = \sum_{n=1}^{N} \frac{T^{n-1}}{\lambda^n} \in B(V).$$

Write $\theta = \|T\|/|\lambda| < 1$. Since, for $1 \leq M \leq N$,

$$\|S_N - S_M\| = \left\| \sum_{n=M+1}^{N} \frac{T^{n-1}}{\lambda^n} \right\| \leq \sum_{n=M+1}^{N} \frac{1}{|\lambda|}\theta^{n-1} = \frac{1}{|\lambda|}\frac{\theta^M - \theta^N}{1-\theta},$$

we see that S_N is a Cauchy sequence. Since $B(V)$ is a Banach space, S_N converges to some $S \in B(V)$, as $N \to +\infty$. By direct claculation,

$$(\lambda I - T)S_N = S_N(\lambda I - T) = I - \lambda^{-N}T^N,$$

and letting $N \to +\infty$ gives that

$$(\lambda I - T)S = S(\lambda I - T) = I.$$

Thus $(\lambda I - T)$ is invertible and, in particular, $\lambda \notin \text{spec}(T)$.

Claim: spec(T) is closed. Choose $\lambda \notin \text{spec}(T)$, then there exists $S \in B(V)$ such that $S(\lambda I - T) = (\lambda I - T)S = I$. Choose $\epsilon < 1/||S||$, then we shall show that

$$\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \cap \text{spec}(T) = \varnothing,$$

i.e., $\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \subset \mathbb{C}\backslash\text{spec}(T)$. If $|\mu - \lambda| < \epsilon$ then, since $(\lambda I - T)S = I$,

$$(\mu I - T) = (\lambda I - T) + (\mu - \lambda)I = (\lambda I - T)(I + (\mu - \lambda)S).$$

However,

(i) $(\lambda I - T)$ is invertible by assumption;
(ii) $(I + (\mu - \lambda)S)$ is invertible since (see Exercise below)

$$\|(\mu - \lambda)S\| = |\mu - \lambda|\|S\| < \epsilon\|S\| < 1.$$

So we have that the product $(\lambda I - T)(I + (\mu - \lambda)S)$ is invertible, i.e., $(\mu I - T)$ is invertible. This shows that $\mu \notin \text{spec}(T)$, as required. Therefore, $\mathbb{C}\backslash\text{spec}(T)$ is open and so $\text{spec}(T)$ is closed. □

Exercise. Let $T \in B(V)$. Show that if $\|T\| < 1$ then $I - T$ is invertible. (Hint: Show that $I + T + \cdots + T^n$, $n \geq 1$ is a Cauchy sequence in $B(V)$ and hence has a limit.)

Solution. This follows from the argument used above to show that $\text{spec}(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$ with $\lambda = 1$.

Remark. It is also true that $\text{spec}(T) \neq \varnothing$. To see this, one needs to consider the complex function

$$\mathbb{C} \to B(V) : \lambda \mapsto \sum_{n=1}^{\infty} \frac{T^{n-1}}{\lambda^n} = (\lambda I - T)^{-1}.$$

There is a theory of analytic functions of a complex variable which take values in Banach spaces. If $\text{spec}(T) = \varnothing$ then one can show that the above function is analytic and bounded. As for complex valued analytic (= differentiable = holomorphic) functions of a complex variable, this forces the function to be constant. (Recall Liouville's Theorem.) One can also show that $\|(\lambda I - T)^{-1}\| \to 0$, as $|\lambda| \to +\infty$, so the constant must be zero. Clearly, this is impossible.

Exercise. Let $V = l^p$, $1 \leq p \leq \infty$, and let T be the shift operator $T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. We know that $\|T\| = 1$, so $\text{spec}(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. Show that $\text{spec}(T) = \{z \in \mathbb{C} : |z| \leq 1\}$. (Hint: show that any $\lambda \in \{z \in \mathbb{C} : |z| < 1\}$ (note $<$) is an eigenvalue of T,

Solution. Suppose that $|\lambda| < 1$. To see that λ is an eigenvalue of T, we shall find a non-zero $(x_1, x_2, x_3, \dots) \in l^p$ such that

$$(x_2, x_3, x_4, \dots) = T(x_1, x_2, x_3, \dots) = \lambda(x_1, x_2, x_3, \dots).$$

Looking at each co-ordinate in turn, we find the conditions

$$x_2 = \lambda x_1,\ x_3 = \lambda x_2 = \lambda^2 x_1,\ x_4 = \lambda x_3 = \lambda^3 x_1,\ \dots,\ x_n = \lambda x_{n-1} = \lambda^{n-1} x_1, \dots .$$

There is no condition on x_1, so let us choose $x_1 = 1$. Then $(1, \lambda, \lambda^2, \dots)$ has the reqired property:

$$T(1, \lambda, \lambda^2, \dots) = (\lambda, \lambda^2, \lambda^3, \dots) = \lambda(1, \lambda, \lambda^2, \dots).$$

(Note that, apart from the choice of x_1, every eigenvector for λ has this form.) Furthermore, since $|\lambda| < 1$,

$$\sum_{i=1}^{\infty} |\lambda^{i-1}|^p = \sum_{i=0}^{\infty} |\lambda^p|^i < +\infty,$$

so $(1, \lambda, \lambda^2, \dots) \in l^p$, $1 \leq p < \infty$ and

$$\sup_{i \geq 1} |\lambda^{i-1}| \leq 1 < +\infty,$$

so $(1, \lambda, \lambda^2, \dots) \in l^\infty$.

Since eigenvalues are contained in the spectrum, this shows that

$$\{z \in \mathbb{C} : |z| < 1\} \subset \text{spec}(T).$$

Since $\text{spec}(T)$ is closed,

$$\{z \in \mathbb{C} : |z| \leq 1\} = \overline{\{z \in \mathbb{C} : |z| < 1\}} \subset \text{spec}(T).$$

On the other hand, $\|T\| = 1$ (†), so, by Proposition 4.7,

$$\text{spec}(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}.$$

Hence

$$\text{spec}(T) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

(†) $\|T\| = 1$ wasn't shown for l^∞ on page 2 but the proof is an easy adaptation of that for l^p, $1 \leq p < \infty$.

Now restrict to $1 \leq p < \infty$ (i.e. don't allow $p = \infty$). If $|\lambda| = 1$ then λ is *not* an eigenvalue of T. By the calculations above, the only possible eigenvectors are constant multiples of

$$(1, \lambda, \lambda^2, \dots).$$

However, since $|\lambda| = 1$,

$$\sum_{i=1}^{\infty} |\lambda^{i-1}|^p = 1 + 1 + 1 + \cdots = +\infty,$$

so

$$(1, \lambda, \lambda^2, \dots) \notin l^p.$$

Thus, no non-zero $x \in l^p$ satisfies $Tx = \lambda x$ and λ is not an eigenvalue. This shows that the spectrum of an operator may contain numbers which are not eigenvlaues.

Given $T : V \to V$, we can consider powers $T^2, T^3, \dots, T^n, \dots$. We can also form polynomial combinations: if $P(x) = a_n x^n + \cdots + a_1 x + a_0$ then we write $P(T) = a_n T^n + \cdots + a_1 T + a_0 I$.

Proposition 4.8. *If $P(x)$ is a polynomial then*

$$\textit{spec}(P(T)) = \{P(\lambda) : \lambda \in \textit{spec}(T)\}.$$

Proof. Suppose P has degree n. For a fixed $\lambda \in \mathbb{C}$, we can write

$$\lambda - P(z) = a(\beta_1 - z)(\beta_2 - z) \cdots (\beta_n - z), \tag{$*$}$$

where $\beta_1, \dots, \beta_n \in \mathbb{C}$ are the roots of the polynomial $z \mapsto \lambda - P(z)$. We can then write

$$\lambda I - P(T) = a(\beta_1 I - T)(\beta_2 I - T) \cdots (\beta_n I - T).$$

If $\lambda \in \operatorname{spec}(P(T))$ then $\lambda I - P(T)$ is not invertible, so $(\beta_i I - T)$ is not invertible for some i, giving $\beta_i \in \operatorname{spec}(T)$. Substituting $z = \beta_i$ in $(*)$, we have $\lambda = P(\beta_i)$. This shows that $\operatorname{spec}(P(T)) \subset \{P(\lambda) : \lambda \in \operatorname{spec}(T)\}$.

Now suppose that $\lambda \notin \operatorname{spec}(P(T))$. Then $(\lambda I - P(T))$ is invertible, so $(\beta_i I - T)$ is invertible for all $i = 1, \dots, n$, i.e., $\{\beta_1, \dots, \beta_n\} \cap \operatorname{spec}(T) = \varnothing$. Since the equation $\lambda - P(z) = 0$ has no other solutions, this shows that $\operatorname{spec}(P(T))^c \cap \{P(\lambda) : \lambda \in \operatorname{spec}(T)\} = \varnothing$. This completes the proof.

Since $\operatorname{spec}(T) \subset \mathbb{C}$ is bounded, we can make the following definition.

Definition. We define the *spectral radius* of T to be the number

$$\rho(T) = \sup\{|\lambda| : \lambda \in \operatorname{spec}(T)\}.$$

By Proposition 4.7, we know that $\rho(T) \leq \|T\|$. A stronger result is given below.

Theorem 4.9 (Spectral Radius Theorem).

$$\rho(T) = \lim_{n\to+\infty} \|T^n\|^{1/n}.$$

Idea of Proof. Since $\operatorname{spec}(T^n) = \{\lambda^n : \lambda \in \operatorname{spec}(T)\}$, for each $n \geq 1$, we have

$$\rho(T)^n = \rho(T^n) \leq \|T^n\|,$$

i.e., $\rho(T) \leq \|T^n\|^{1/n}$. Therefore, $\rho(T) \leq \liminf_{n\to+\infty} \|T^n\|^{1/n}$.

The proof of the lower bound is omitted. (It requires the theory of analytic Banach space valued functions.) □

Spectra of Operators on Hilbert Spaces

Let $T : H \to H$ be a bounded linear operator on a Hilbert space H and let $T^* : H \to H$ be its adjoint operator.

Lemma 4.10.

(i) *$\operatorname{spec}(T^*) = \overline{\operatorname{spec}(T)}$, i.e., $\lambda \in \operatorname{spec}(T) \iff \bar{\lambda} \in \operatorname{spec}(T^*)$;*

(ii) *if T^{-1} exists then $\operatorname{spec}(T^{-1}) = (\operatorname{spec}(T))^{-1} = \{\lambda^{-1} : \lambda \in \operatorname{spec}(T)\}$, i.e., $\lambda \in \operatorname{spec}(T) \iff \lambda^{-1} \in \operatorname{spec}(T^{-1})$.*

Proof.
(i) We have$\lambda \notin \overline{\operatorname{spec}(T)}$ if and only if $S = (\bar{\lambda}I - T)$ is invertible. Since $SS^{-1} = I \iff I = I^* = (SS^{-1})^* = (S^{-1})^*S^*$ and $S^{-1}S = I \iff I = I^* = (S^{-1}S)^* = S^*(S^{-1})^*$, $S^* = (\lambda I - T^*)$ is invertible if and only if $(\bar{\lambda}I - T)$ is invertible.

(ii) If $\lambda \notin \operatorname{spec}(T)$ then $S(\lambda I - T) = I = (\lambda I - T)S$, where $S = (\lambda I - T)^{-1}$. Notice that $ST = -I + \lambda S = TS$. Thus if $U = -\lambda ST = -\lambda TS$ then $U(\lambda^{-1}I - T^{-1}) = I = (\lambda^{-1}I - T^{-1})U$, so $(\lambda^{-1}I - T^{-1})$ is invertible, i.e., $\lambda^{-1} \notin \operatorname{spec}(T^{-1})$. By symmetry, $\operatorname{spec}(T^{-1}) = \{\lambda^{-1} : \lambda \in \operatorname{spec}(T)\}$.

Definition. We say that $T : H \to H$ is *normal* if $TT^* = T^*T$. (Notice that if T is self-adjoint then T is normal.)

Suppose that $T^*T = I$. Then (and only then) we have, for all $x \in H$,

$$\|x\|^2 = \langle x, x\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = \|Tx\|^2.$$

So T is an isometry if and only if $T^*T = I$.

Definition. We say that T is unitary if $T^* = T^{-1}$. (Note that if T is unitary then T is an isometry. Conversely, if T is an isometry and T is invertible then T is unitary.)

Theorem 4.11.

(i) *If T is normal then $\rho(T) = \|T\|$.*

(ii) *If T is an isometry then $\rho(T) = 1$.*

(iii) *If T is unitary then $\operatorname{spec}(T) \subset \{z \in \mathbb{C} : |z| = 1\}$.*

(iv) *It T is self-adjoint then $\operatorname{spec}(T) \subset [-\|T\|, \|T\|] \subset \mathbb{R}$.*

Proof. (i) For $n > 0$,

$$\begin{aligned}\|T^{2^n}\|^2 &= \|(T^{2^n})^*(T^{2^n})\| = \|(T^*T)^{2^n}\| \\ &\text{(by normality } T^*T = TT^*) \\ &= \|(T^*T)^{2^{n-1}}\|^2 \\ &\text{(since } S = (T^*T)^{2^{n-1}} \text{ satisfies } \|S^*S\| = \|S\|^2) \\ &= \|(T^*T)^{2^{n-2}}\|^4 = \cdots = \|T^*T\|^{2^n} = \|T\|^{2^{n+1}}.\end{aligned}$$

So $\|T^{2^n}\| = \|T\|^{2^n}$. Thus $\rho(T) = \lim_{n\to+\infty} \|T^{2^n}\|^{1/2^n} = \|T\|$.

(ii) We have $\|T^n\|^2 = \|(T^*)^n T^n\|$. Repeatedly using $T^*T = I$, we get $\|T^n\|^2 = \|I\| = 1$. Hence $\rho(T) = \lim_{n\to+\infty} \|T^n\|^{1/n} = 1$.

(iii) If T is unitary then T is an isometry, so by (ii), $\text{spec}(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. Thus by Lemma 5.5, $\text{spec}(T^{-1}) \subset \{z \in \mathbb{C} : |z| \geq 1\}$. But $\text{spec}(T^{-1}) = \text{spec}(T^*) = \{\bar{z} : z \in \text{spec}(T)\}$ and $|\bar{z}| = |z|$, so this forces $\text{spec}(T) \subset \{z \in \mathbb{C} : |z| = 1\}$.

(iv) If T is self-adjoint then T is normal so, by (i), $\rho(T) = \|T\|$. Let $\alpha \in \mathbb{C}$ and consider $(\alpha I - T)$. It will suffice to show that $(\alpha I - T)$ is invertible whenever $\Im\alpha \neq 0$. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < 1/\|T\|$. Then $(I + i|\lambda|T)^{-1}$ exists (since $|\lambda|\|T\| < 1$). Let $U = (I - i|\lambda|T)(I + i|\lambda|T)^{-1}$. Then we also have $U = (I + i|\lambda|T)^{-1}(I - i|\lambda|T)$.
Claim: U is unitary. We shall use the facts that $(S^*)^{-1} = (S^{-1})^*$ and $(zS)^* = \bar{z}S^*$. We have

$$\begin{aligned}U^{-1} &= (I + i|\lambda|T)(I - i|\lambda|T)^{-1} \\ &= (I + i|\lambda|T^*)(I - i|\lambda|T^*)^{-1} \\ &= (I - i|\lambda|T)^*((I + i|\lambda|T)^*)^{-1} \\ &= (I - i|\lambda|T)^*((I + i|\lambda|T)^{-1})^* \\ &= ((I + i|\lambda|T)^{-1}(I - i|\lambda|T))^* = U^*.\end{aligned}$$

Now, if $\Im\alpha \neq 0$ then

$$\left|\frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha}\right| \neq 1,$$

so, by (iii),

$$\left(\frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha}\right) I - U$$

is invertible. However,

$$\begin{aligned}&\left(\frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha}\right) I - U \\ &= \frac{1}{1 + i|\lambda|\alpha}\left((1 - i|\lambda|\alpha)(I + i|\lambda|T) - (I - i|\lambda|T)(1 + i|\lambda|\alpha)\right)(I + i|\lambda|T)^{-1} \\ &= \frac{-2i|\lambda|}{1 + i|\lambda|\alpha}(\alpha I - T)(I + i|\lambda|T)^{-1}.\end{aligned}$$

Since the L.H.S. is invertible and $(I + i|\lambda|T)^{-1}$ is invertible, this shows that $(\alpha I - T)$ is invertible, as required. $\square$