MATH31001/MATH41001/MATH61001: LINEAR ANALYSIS

CHAPTER 3: LINEAR FUNCTIONALS

CONTINUOUS FUNCTIONALS

Definition. Let V be a (normed) vector space over $\mathbb{R}$ (or $\mathbb{C}$). A *linear functional* on V is a map $f : V \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

for all $x, y \in V$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

Example. Let $V = \mathbb{R}^n$. Then every linear functional $f : \mathbb{R}^n \to \mathbb{R}$ has the form

$$f(x) = \sum_{i=1}^{n} a_i x_i = \langle a, x \rangle,$$

where $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$.

Example. Let $V = C([0,1], \mathbb{R})$. Then an example of a linear functional $f : C([0,1], \mathbb{R}) \to \mathbb{R}$ is given by

$$f(\phi) = \int_0^1 \phi(x) dx.$$

Definition. The *dual space* of V, denoted by V^*, is the set of all *continuous* linear functionals on V, i.e., those for which $\lim_{n\to+\infty} \|x_n - x\| = 0$ implies $\lim_{n\to+\infty} f(x_n) = f(x)$. (We shall see later that V^* is itself a vector space.)

Definition. A linear functional f is called *bounded* if there exists $M \geq 0$ such that

$$|f(x)| \leq M\|x\|, \quad \text{for all } x \in V.$$

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Proposition 3.1. *Let $f : V \to \mathbb{R}$ (or $\mathbb{C}$) be a linear functional on a normed vector space V. Then f is continuous if and only if f is bounded.*

Proof.
($\Longrightarrow$) Suppose that f is continuous. Assume (for a contradiction) that there is no $M \geq 0$ for which $|f(x)| \leq M\|x\|$, for all $x \in V$. Then we can choose a sequence $x_n \in V$, $n \geq 1$, such that $|f(x_n)| > n\|x_n\|$, so that

$$\left| f\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right)\right| = \frac{|f(x_n)|}{n\|x_n\|} > 1.$$

On the other hand,

$$\left\|\frac{1}{n}\frac{x_n}{\|x_n\|}\right\| \to 0, \text{ as } n \to +\infty,$$

so, by continuity at 0,

$$f\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right) \to f(0) = 0, \text{ as } n \to +\infty,$$

giving the required contradiction.
($\Longleftarrow$) Suppose that f is bounded. Given $x \in V$ and $\epsilon > 0$, we need to show that there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. If $M = 0$ then $|f(x) - f(y)| = |f(x - y)| = 0$, so any δ will do. If $M > 0$, choose $\delta = \epsilon/M$. Then, if $\|x - y\| < \delta$,

$$|f(x) - f(y)| = |f(x - y)| \leq M\|x - y\| < M\frac{\epsilon}{M} = \epsilon,$$

as required. $\square$

Definition. Let V be a normed vector space (with norm $\|\cdot\|$). Then we define a norm on V^* by

$$\|f\| = \sup_{\|x\|=1} |f(x)|.$$

By Proposition 3.1, this is finite and an equivalent definition is

$$\|f\| = \sup_{\|x\|\neq 0} \frac{|f(x)|}{\|x\|}.$$

(N.B. it still has to be proved that this is a norm – we do this in Proposition 3.2.)

An immediate consequence of the definition is the following estimate. We shall use it frequently.

Corollary 3.1.2. *For all $x \in V$,*

$$|f(x)| \leq \|f\| \|x\|.$$

Example. $V = C([0,1], \mathbb{R})$ (with $\| \cdot \|_\infty$),

$$f(\phi) = \int_0^1 \phi(x) dx.$$

Then

$$|f(\phi)| \leq \|\phi\|_\infty \quad \text{so} \quad \|f\| \leq 1.$$

Putting $\phi(x) = 1 \; \forall x \in [0,1]$ gives

$$\|f\| \geq |f(1)| = 1.$$

Thus $\|f\| = 1$.

Example. $V = l^1$ (with $\| \cdot \|_1$),

$$f((x_i)_{i=1}^\infty) = x_1.$$

Then

$$|f((x_i)_{i=1}^\infty)| = |x_1| \leq \sum_{i=1}^\infty |x_i| = \|(x_i)_{i=1}6\infty)\|_1 \quad \text{so} \quad \|f\| \leq 1.$$

Putting $(x_i)_{i=1}^\infty = (1, 0, 0, \dots)$ gives

$$\|f\| \geq |f((1, 0, 0, \dots))| = 1.$$

Thus $\|f\| = 1$.

Proposition 3.2. *If $(V, \| \cdot \|)$ is a normed vector space then $(V^*, \| \cdot \|)$ is a Banach space.*

Proof.
V^* is a vector space: Suppose that $f, g \in V^*$ and that λ is a scalar. Then

$$(\lambda f)(x) = \lambda f(x)$$

and

$$(f + g)(x) = f(x) + g(x)$$

and these are clearly continuous.
$\| \cdot \|$ is a norm:

(1) Clearly $\|f\| \geq 0$ and

$$\|f\| = 0 \quad \Longleftrightarrow \quad \sup_{\|x\|=1} |f(x)| = 0 \quad \Longleftrightarrow \quad |f(x)| = 0, \text{ for all } x \text{ with } \|x\| = 1$$

and the latter identity is equivalent to $f = 0$.

(2)
$$\|\lambda f\| = \sup_{\|x\|=1} |\lambda||f(x)| = |\lambda| \sup_{\|x\|=1} |f(x)| = |\lambda|\|f\|.$$

(3) For $\|x\| = 1$,
$$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq \sup_{\|x\|=1} |f(x)| + \sup_{\|x\|=1} |g(x)| = \|f\| + \|g\|. \end{aligned}$$

Taking the supremum, we get
$$\|f + g\| = \sup_{\|x\|=1} |f(x) + g(x)| \leq \|f\| + \|g\|.$$

V^* is a Banach space: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in V^*. Fixing $x \in V$,
$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|\|x\|,$$
so $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) and we may write $f(x) = \lim_{n\to+\infty} f_n(x)$. We need to show that this f is an element of V^*.

First we check that f is linear:
$$f(\lambda x + \mu y) = \lim_{n\to+\infty} f_n(\lambda x + \mu y) = \lim_{n\to+\infty} (\lambda f_n(x) + \mu f_n(y)) = \lambda f(x) + \mu f(y).$$

Next, we check that f is bounded (remember this is equivalent to continuous). Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, we may choose $N \geq 1$ so that $n, m \geq N$ implies that $\|f_n - f_m\| \leq 1$. We have, for $\|x\| = 1$ and $n, m \geq N$,
$$\begin{aligned} |f(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \\ &= \lim_{n\to+\infty} |f_n(x) - f_N(x)| + |f_N(x)| \\ &\leq \limsup_{n\to+\infty} \|f_n - f_N\| + \|f_N\| \leq 1 + \|f_N\|, \end{aligned}$$
so f is bounded, as required.

To finish, we check that f_n converges to f in the norm $\|\cdot\|$ on V^*. Using the fact that, for each $x \in V$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence, given $\epsilon > 0$, we may choose $N \geq 1$ such that $n, m \geq N$ implies that
$$|f_n(x) - f_m(x)| \leq \epsilon.$$
Letting $m \to +\infty$ gives that, for $n \geq N$,
$$|f_n(x) - f(x)| \leq \epsilon,$$
so that
$$\|f_n - f\| = \sup_{\|x\|=1} |f_n(x) - f(x)| \leq \epsilon.$$
In other words,
$$\lim_{n\to+\infty} \|f_n - f\| = 0.$$ □

Remark. Note that we did not need to assume that V is a Banach space.

Example 1. Let H be a Hilbert space (with inner product $\langle\cdot,\cdot\rangle$) and choose $y \in H$. Then $f_y(x) = \langle x, y\rangle$ is a bounded (continuous) linear functional. We have

$$\|f_y\| = \sup_{x\neq 0} \frac{\langle x, y\rangle}{\|x\|}.$$

By the Cauchy-Schwartz inequality,

$$\frac{\langle x, y\rangle}{\|x\|} \leq \|y\|,$$

with equality for $x = y$. Therefore, $\|f_y\| = \|y\|$.

Example 2. Consider l^p for $1 < p < +\infty$ and define $1 < q < +\infty$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Put $a = (a_1, a_2, \dots) \in l^p$ and $b = (b_1, b_2, \dots) \in l^q$ (b fixed). Define

$$f_b(a) = \sum_{i=1}^{\infty} a_i b_i.$$

Lemma 3.3. *$f_b \in (l^p)^*$ and $\|f_b\| = \|b\|_q$.*

In the case $p = q = 2$, we discussed this result in Example 1 (since l^2 is a Hilbert space).

Proof. First note that

$$f_b(a) = \sum_{i=0}^{\infty} a_i b_i$$

is finite since, by Hölder's inequality,

$$|f_b(a)| \leq \sum_{i=1}^{\infty} |a_i b_i| \leq \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |b_i|^q\right)^{1/q} = \|a\|_p \|b\|_q. \qquad (*)$$

Clearly f_b is linear and $(*)$ shows that f is bounded and, furthermore, that $\|f_b\| \leq \|b\|_q$.

To show equality, we shall make a particular choice for a, depending on b. If $b_i = |b_i|e^{i\theta}$ then we choose $a_i = |b_i|^{q/p} e^{-i\theta}$. Then $a_i b_i = |a_i b_i|$ and $|a_i|^p/|b_i|^q = 1$. Thus

$$\|f_b\|\|a\|_p \geq |f_b(a)| = \sum_{i=1}^{\infty} a_i b_i = \|a\|_p \|b\|_q$$

(using the equality statement in Hölder's inequality). Therefore $\|f_b\| = \|b\|_q$. $\square$

Example 3. Consider $l^1 = \{a = (a_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |a_i| < +\infty,\ a_i \in \mathbb{C}\}$. Fix $b = (b_i)_{i=1}^{\infty} \in l^{\infty}$ and define

$$f_b(a) = \sum_{i=0}^{\infty} a_i b_i.$$

Lemma 3.4. *$f_b \in (l^1)^*$ and $\|f_b\| = \|b\|_\infty$.*

Proof. The sum is finite since

$$\left|\sum_{i=1}^{\infty} a_i b_i\right| \le \sum_{i=1}^{\infty} |a_i b_i| \le \left(\sum_{i=1}^{\infty} |a_i|\right)\left(\sup_{i\ge 1} |b_i|\right) = \|a\|_1 \|b\|_\infty.$$

Clearly f_b is linear and the above ineqality also shows that $\|f_b\| \le \|b\|_\infty$.

To show equality, we shall make a particular choices for a, depending on b. Given $\epsilon > 0$, choose $j \ge 1$ so that $|b_j| > \|b\|_\infty - \epsilon$. If $b_j = |b_j| e^{i\theta}$ then we choose

$$a_i = \begin{cases} e^{-i\theta} & \text{if } i = j \\ 0 & \text{of } i \ne j. \end{cases}$$

Then $a_j b_j = |a_j b_j| = |b_j|$ and $\|a\|_1 = |a_j| = 1$. Thus

$$\|f_b\| = \|f_b\| \|a\|_1 \ge |f_b(a)| = \sum_{i=1}^{\infty} a_i b_i = a_j b_j > \|b\|_\infty - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this gives us $\|f_b\| \ge \|b\|_\infty$, as required. ☐

LINEAR FUNCTIONALS ON HILBERT SPACES

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We showed earlier that, given $x \in H$, we can define a linear functional $f_x \in H^*$ by

$$f_x(y) = \langle y, x \rangle,$$

for all $y \in H$, and that $\|f_x\| = \|x\|$. We shall now show that every element of H^* can be written in this form.

More formally, we can think of the above construction as a (linear) map

$$H \to H^* : x \mapsto f_x.$$

Definition. A linear map $T : V \to V'$ between two normed vector spaces $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ is called an *isometric isomorphism* if it is a bijection and satisfies $\|Tx\|' = \|x\|$, for all $x \in V$. We then write $V = V'$.

Theorem 3.5 (Riesz Representation Theorem). *The linear map*

$$H \to H^* : x \mapsto f_x$$

is an isometric isomorphism.

Proof. We have already seen that $\|f_x\| = \|x\|$. Furthermore, if $x \ne y$ then $\|f_x - f_y\| = \|f_{x-y}\| = \|x - y\| > 0$, so $x \mapsto f_x$ is injective. Thus it only remains to show that $x \mapsto f_x$ is surjective.

Choose $f \in H^*$, $f \neq 0$, and let

$$L = \ker f = \{y \in H \,:\, f(y) =\}.$$

Since $f \neq 0$, we have $L \neq H$, so $L^\perp \neq \{0\}$. Thus we can choose $x_0 \in L^\perp$ (so, in particular, $f(x_0) \neq 0$).

Given $y \in H$, let

$$w = y - \frac{f(y)}{f(x_0)} x_0.$$

Then

$$f(w) = f\left(y - \frac{f(y)}{f(x_0)} x_0\right) = 0,$$

so $w \in L$. However, since $x_0 \in L^\perp$, this gives

$$0 = \langle w, x_0 \rangle = \langle y, x_0 \rangle - \langle x_0, x_0 \rangle \frac{f(y)}{f(x_0)},$$

i.e.,

$$\langle y, x_0 \rangle = \langle x_0, x_0 \rangle \frac{f(y)}{f(x_0)}.$$

Rearranging, we get

$$f(y) = \left\langle y, \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0 \right\rangle = f_v(y),$$

where

$$v = \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0.$$

So, $f = f_v$ and we see that $x \mapsto f_x$ is surjective. □

Remark. We may express the above result more succinctly as $H = H^*$.

Theorem 3.6. *(i) Suppose that* $1 < p, q < +\infty$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then the linear map

$$l^q \to (l^p)^* : b \mapsto f_b$$

defined above is an isometric isomorphism, so $(l^p)^* = l^q$.
(ii) The linear map

$$l^\infty \to (l^1)^* : b \mapsto f_b$$

defined above is an isometric isomorphism, so $(l^1)^* = l^\infty$.

Proof. Omitted. **(Extra reading for MATH41001/MATH61001.)** □

THE HAHN-BANACH THEOREM

The next result is about linear functionals on subspaces. It will turn out to be useful when we want to construct functionals with particular properties. Roughly, one can define a bounded linear functional on some subspace of a larger Banach space and then know that one can extend it to the whole space without increaing the norm.

Let V be a normed vector space over $\mathbb{R}$ (we deal with complex spaces later) and let $W \subset V$ be a linear subspace. Then, given $\widetilde{f} \in V^*$, we may define a linear functional $f \in W^*$ by restriction, i.e.,

$$f(x) = \widetilde{f}(x), \qquad \text{for } x \in W.$$

The next result gives the converse.

Theorem 3.7 (Hahn-Banach Theorem). *Let V be a normed vector space over $\mathbb{R}$ and let $W \subset V$ be a linear subspace. Suppose that $f \in W^*$ then f can be extended to a linear functional $\widetilde{f} \in V^*$ with $\|\widetilde{f}\| = \|f\|$. (Here, "extended" means that $\widetilde{f}(x) = f(x)$ for $x \in W$.)*

We now state the Hahn-Banach Theorem for complex vector spaces.

Theorem 3.8 (Complex Hahn-Banach Theorem). *Let V be a normed vector space over $\mathbb{C}$ and let $W \subset V$ be a linear subspace. Suppose that $f \in W^*$ then f can be extended to a linear functional $\widetilde{f} \in V^*$ with $\|\widetilde{f}\| = \|f\|$.*

(The proof of these theorems is extra reading for MATH41001/MATH61001.)

THE SECOND DUAL

Definition. Since V^* is itself a normed vector space, it has a dual space $(V^*)^*$. We call this the *second dual* of V and write $V^{**} = (V^*)^*$.

There is a natural way in which we may regard V a subset of V^{**}. For each $x \in V$, define a function $i(x) : V^* \to \mathbb{C}$ by

$$i(x)(f) = f(x),$$

where $f \in V^*$. It is easy to see that $i(x)$ is linear.

Lemma 3.9. $\|i(x)\| = \|x\|$.

Proof. For any $f \in V^*$,

$$|i(x)(f)| = |f(x)| \leq \|f\|\|x\|,$$

so $i(x)$ is bounded and

$$\|i(x)\| = \sup_{\|f\|=1} |i(x)(f)| \leq \|x\|.$$

Now we shall get a bound in the other direction by choosing a linear functional f such that $i(x)(f) = \|x\|$. To do this, we use the Hahn-Banach Theorem. Let W denote the vector space spanned by x. Define $f_0 \in W^*$ by $f(\lambda x) = \lambda\|x\|$, which has norm

$$\|f_0\| = \sup_{\lambda \neq 0} \frac{|f(\lambda x)|}{|\lambda|\|x\|} = 1.$$

By the Hahn-Banach Theorem, we can choose $f \in V^*$ such that (in particular) $f(x) = \|x\|$ and $\|f\| = \|f_0\| = 1$. Then

$$\|i(x)\| = \|i(x)\|\|f\| \geq |i(x)(f)| = |f(x)| = \|x\|,$$

as required. $\square$

Remark. Compare the second part of this proof to the second parts of the proofs of Lemmas 3.3 and 3.4. There, we could make choices (for a depending on b) using the particular properties of the spaces, without having to use the Hahn-Banach Theorem.

By Lemma 3.9, we have defined a (linear) map $i : V \to V^{**}$ and it is easy to see that i is injective. We can think of $i(V)$ as a copy of V sitting inside V^{**}.

Definition. We say that a normed vector space V is *reflexive* if $i : V \to V^{**}$ is an isometric isomorphism. In view of the above discussion, one only needs to check that $i : V \to V^{**}$ is a surjection.

Theorem 3.10. *Every Hilbert space H is reflexive.*

Proof. Let $T_H : H \to H^* : x \mapsto f_x$ be the isometric isomorphism discussed above. Since H^* is a Hilbert space, there is a corresponding isometric isomorphism $T_{H^*} : H^* \to H^{**}$. By the Riesz Representation Theorem, any $f \in H^*$ has the form $f = f_x = T_H(x)$, for some unique $x \in H$, and any $\psi \in H^{**}$ has the form $\psi = T_{H^*}(T_H(y))$, for some unique $y \in H$. We have

$$i(y)(f) = f(y) = \langle x, y \rangle = \langle T_H x, T_H y \rangle = \langle f, T_H y \rangle = T_{H^*}(T_H(y))(f) = \psi(f),$$

so that i is surjective. $\square$

Example. Consider l^p for $1 < p < +\infty$. By Theorem 3.6,

$$(l^p)^{**} = (l^q)^* = l^p$$

(where $p^{-1} + q^{-1} = 1$) and l^p is reflexive.

Example. By Theorem 3.6, $(l^1)^* = l^\infty$. However, $(l^\infty)^* \neq l^1$ (hard) and l^1 is not reflexive.