

MATH 31001/41001 Linear Analysis

Solution Sheet 7

1.(i) For any $x, y \in H$, we have

$$\langle STx, y \rangle = \langle Tx, S^*y \rangle = \langle x, T^*S^*y \rangle.$$

This shows that $(ST)^* = T^*S^*$.

(ii) Notice that, for any $x \in H$,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle \leq \|x\| \|T^*Tx\| \leq \|T^*T\| \cdot \|x\|^2$$

(where we have used the Cauchy-Schwarz inequality in the penultimate step), so that

$$\|T\|^2 \leq \|T^*T\|.$$

For the reverse inequality,

$$\|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$$

(since $\|T^*\| = \|T\|$). Thus $\|T^*T\| = \|T\|^2$. To show $\|TT^*\| = \|T\|^2$, replace T by T^* then $\|T^*\| = \|T\|$ and $T^{**} = T$ gives the result.

2. For $x = (x_1, x_2, x_3, \dots) \in \ell^2$, we have

$$\|T_n(x)\|_2 = |x_n| \leq \|x\|_2.$$

This shows that $\|T_n\| \leq 1$.

Now for each n , we have

$$\|T_n(e_n)\| = 1 = \|e_n\|_2,$$

so that $\|T_n\| \geq 1$. This shows that $\|T_n\| = 1$.

(i) Suppose that $(T_1 - \lambda I)(x) = 0$. Then

$$(x_1 - \lambda x_1, -\lambda x_2, -\lambda x_3, \dots) = 0.$$

For $x_1 \neq 0$, $(x_1, 0, 0, \dots)$ gives a solution $\lambda = 1$.

For $x_2 \neq 0$, $(0, x_2, 0, \dots)$ gives a solution $\lambda = 0$.

For $\lambda \neq 0, 1$, $x_1 = \lambda x_1$ forces $x_1 = 0$ and, for $j \geq 2$, $-\lambda x_j = 0$ forces $x_j = 0$. Therefore there are no other eigenvalues for T_1 .

(b) For $n \geq 2$, suppose that $(T_n - \lambda I)(x) = 0$. Then

$$(x_n - \lambda x_1, -\lambda x_2, -\lambda x_3, \dots) = 0.$$

For $x_1 \neq 0$, $(x_1, 0, 0, \dots)$ gives a solution $\lambda = 0$.

For $\lambda \neq 0$, $-\lambda x_j = 0$ forces $x_j = 0$, for $j \geq 2$. In particular, $x_n = 0$. Then $x_n = \lambda x_1$ forces $x_1 = 0$. Therefore there are no other eigenvalues for T_n , $n \geq 2$.

3. (i) Clearly, $T(e_n) = a_n e_n$, so $a_n \in \text{spec}(T)$, for all $n \geq 1$.

(ii) If $\lambda \notin \overline{\{a_i : i \geq 1\}}$ then there exists $\epsilon > 0$ such that $|\lambda - a_i| \geq \epsilon$, for all $i \geq 1$. We have

$$(\lambda I - T)((x_i)_{i=1}^\infty) = ((\lambda - a_i)x_i)_{i=1}^\infty,$$

which has inverse

$$S((x_i)_{i=1}^\infty) = ((\lambda - a_i)^{-1}x_i)_{i=1}^\infty.$$

Since

$$|(\lambda - a_n)^{-1}| \leq 1/\epsilon,$$

S is a bounded linear operator (with $\|S\| \leq 1/\epsilon$).

By (i) and (ii), we have

$$\{a_n : n \geq 0\} \subset \text{spec}(T) \subset \overline{\{a_n : n \geq 0\}}.$$

Since $\text{spec}(T)$ is closed, we have $\text{spec}(T) = \overline{\{a_n : n \geq 0\}}$.

4. From the definition of T , $T((x_i)_{i=1}^\infty) = \left(\frac{x_{i+1}}{i}\right)_{i=1}^\infty$, $T^2((x_i)_{i=1}^\infty) = \left(\frac{x_{i+2}}{i(i+1)}\right)_{i=1}^\infty$, and by induction,

$$T^n((x_i)_{i=1}^\infty) = \left(\frac{x_{i+n}}{i(i+1)\cdots(i+n-1)}\right)_{i=1}^\infty.$$

Thus

$$\|T^n((x_i)_{i=1}^\infty)\|_2^2 = \sum_{i=1}^{\infty} |x_{i+n}|^2 \left(\frac{(i-1)!}{(i+n-1)!}\right)^2 \leq \sum_{i=1}^{\infty} \frac{|x_{i+n}|^2}{(n!)^2} \leq \frac{\|(x_i)_{i=1}^\infty\|_2^2}{(n!)^2}.$$

Hence

$$\|T^n\| \leq \frac{1}{n!}.$$

Now take $(x_i)_{i=1}^\infty = e_n$, then $\|T^n(e_n)\| = 1/n!$. Thus $\|T^n\| = 1/n!$.

To show that $\text{spec}(T) = \{0\}$, we shall calculate the spectral radius $\rho(T)$. We have

$$\rho(T) = \lim_{n \rightarrow +\infty} \|T^n\|^{1/n} = \left(\frac{1}{n!}\right)^{1/n} = 0,$$

using Stirling's formula. Since $\text{spec}(T) \subset \{z \in \mathbb{C} : |z| \leq \rho(T)\}$ and $\text{spec}(T)$ is non-empty, we conclude that $\text{spec}(T) = \{0\}$.

5. (i) For any $x \in X$ and $f \in X^*$, we have

$$|(T^*f)(x)| = |f(Tx)| \leq \|f\| \|Tx\| \leq \|f\| \|T\| \|x\|,$$

which shows that

$$\|T^*\| = \sup_{f \neq 0} \frac{\|T^*f\|}{\|f\|} = \sup_{f \neq 0} \frac{1}{\|f\|} \sup_{x \neq 0} \frac{|(T^*f)(x)|}{\|x\|} \leq \|T\|.$$

(ii) Choose $\epsilon > 0$. By the definition of $\|T\|$, we can choose $x_0 \in X$ such that

$$\|x_0\| = 1 \quad \text{and} \quad \|Tx_0\| \geq \|T\| - \epsilon.$$

Now, using Proposition E.4 (in the 41001/61001 extra reading), we can find $f \in X^*$ such that

$$|f(Tx_0)| = \|Tx_0\| \quad \text{and} \quad \|f\| = 1.$$

Then

$$\|T\| - \epsilon \leq \|Tx_0\| = |(T^*f)(x_0)| \leq \|T^*\|$$

(since $\|x_0\| = 1$ and $\|f\| = 1$). Since $\epsilon > 0$ is arbitrary, we have $\|T^*\| \geq \|T\|$.