MATH31001/41001/61001 Linear Analysis

Solution Sheet 6

1. The fact that T is linear is a straightforward check. Since the map $x \mapsto x^2$ is a bijection on $[0,1]$, we have

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| = \sup_{x\in[0,1]} |f(x^2)| = \|Tf\|_\infty$$

for any $f \in C[0,1]$. Hence $\|T\| = 1$. (In fact, T is an isometry.)

2. It is clear that T is linear since

$$T(\lambda g + \mu h) = f(\lambda g + \mu h) = \lambda(fg) + \mu(fh) = \lambda T(g) + \mu T(h).$$

T is bounded since

$$\|T(g)\|_\infty = \|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$$

and this estimate also gives $\|T\| \le \|f\|_\infty$. Also, if $g = 1$ then $\|T(1)\|_\infty = \|f\|_\infty$, so we have $\|T\| = \|f\|_\infty$.

3. *$B(V, V')$ is a vector space:* Suppose that $T, S \in B(V, V')$ and that λ is a scalar. Then

$$(\lambda T)(x) = \lambda T(x)$$

and

$$(T + S)(x) = T(x) + S(x)$$

and these are clearly continuous (which, remember, is equivalent to bounded).

$\|\cdot\|$ is a norm on $B(V, V')$:

(1) Clearly $\|T\| \ge 0$ and

$$\|T\| = 0 \iff \sup_{\|x\|=1} \|T(x)\| = 0 \iff \|T(x)\| = 0, \text{ for all } x \text{ with } \|x\| = 1$$

and the latter identity is equivalent to $T = 0$.

(2)

$$\|\lambda T\| = \sup_{\|x\|=1} |\lambda| \|T(x)\| = |\lambda| \sup_{\|x\|=1} \|T(x)\| = |\lambda| \|T\|.$$

(3) For $\|x\| = 1$,

$$\begin{aligned}\|T(x) + S(x)\| &\le \|T(x)\| + \|S(x)\| \\ &\le \sup_{\|x\|=1} \|T(x)\| + \sup_{\|x\|=1} \|S(x)\| = \|T\| + \|S\|.\end{aligned}$$

Taking the supremum, we get

$$\|T + S\| = \sup_{\|x\|=1} \|T(x) + S(x)\| \le \|T\| + \|S\|.$$

$B(V,V')$ is a Banach space: Suppose that $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $B(V,V')$. Fixing $x \in V$,

$$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| \cdot \|x\|,$$

so $\{T_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in V'. Since V' is a Banach space (hence complete) this sequence converges and we may write $T(x) = \lim_{n\to+\infty} T_n(x)$. We need to show that this T is an element of $B(V,V')$. First we check that T is linear:

$$T(\lambda x + \mu y) = \lim_{n\to+\infty} T_n(\lambda x + \mu y) = \lim_{n\to+\infty} (\lambda T_n(x) + \mu T_n(y)) = \lambda T(x) + \mu T(y).$$

Next, we check that T is bounded (hence continuous). Since $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence, we may choose $N \geq$ so that $n, m \geq N$ implies that $\|T_n - T_m\| \leq 1$. We have, for $\|x\| = 1$ and $n, m \geq N$,

$$\begin{aligned}\|T(x)\| &\leq \|T(x) - T_N(x)\| + \|T_N(x)\| \\ &= \lim_{n\to+\infty} \|T_n(x) - T_N(x)\| + \|T_N(x)\| \\ &\leq \limsup_{n\to+\infty} \|T_n - T_N\| + \|T_N\| \leq 1 + \|T_N\|,\end{aligned}$$

so T is bounded, as required.

To finish, we check that T_n converges to T in the norm $\|\cdot\|$ on $B(V,V')$. Using the fact that, for each $x \in V$, $\{T_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence, given $\epsilon > 0$, we may choose $N \geq 1$ such that $n, m \geq N$ implies that

$$\|T_n(x) - T_m(x)\| \leq \epsilon.$$

Letting $m \to +\infty$ gives that, for $n \geq N$,

$$\|T_n(x) - T(x)\| \leq \epsilon,$$

so that

$$\|T_n - T\| = \sup_{\|x\|=1} \|T_n(x) - T(x)\| \leq \epsilon.$$

In other words,

$$\lim_{n\to+\infty} \|T_n - T\| = 0.$$

4. The linearity of T_a is clear.

For $x = (x_1, x_2, x_3, \ldots) \in \ell^2$, we have

$$\|T_a(x)\|_2^2 = \sum_{i=1}^{\infty} |a_i x_i|^2 \leq \|a\|_\infty^2 \sum_{i=1}^{\infty} |x_i|^2 = \|a\|_\infty^2 \|x\|_2^2,$$

so that

$$\|T_a(x)\|_2 \leq \|a\|_\infty \|x\|_2.$$

This shows that T_a is bounded.

If a is a real vector, we have

$$\langle x, Ty\rangle = \sum_{i=1}^{\infty} x_i \overline{(a_i y_i)} = \sum_{i=1}^{\infty} (a_i x_i)\overline{y_i} = \langle Tx, y\rangle,$$

for all $x, y \in \ell^2$, so that T is self-adjoint. If a is not real and the entry $a_j \in \mathbb{C}\backslash\mathbb{R}$, say, then choose

$$x = y = e_j,$$

the vector with entry 1 in the jth place and 0 elsewhere. We have

$$\langle x, Ty\rangle = \overline{a_j} \neq a_j \langle Tx, y\rangle,$$

so that T_a is not self-adjoint.

5*. By the Riesz Representation Theorem, we can find $y \in H$ such that

$$f(x) = f_y(x) = \langle x, y\rangle,$$

for all $x \in H$. Similarly, we can find $y' \in H$ such that

$$(T^* f)(x) = f_{y'}(x) = \langle x, y'\rangle,$$

for all $x \in H$.

In particular, we have

$$\langle x, y'\rangle = (T^* f)(x) = f(Tx) = \langle Tx, y\rangle,$$

for all $x \in H$.

Now, the Riesz Representation Theorem says that the map $y \mapsto f_y$ is an isometric isomorphism between H and H^*, so we can identify y with f and y' with $T^* f$. Thus, if we define an operator (which we still call the adjoint) $T^* : H \to H$ by $T^*(y) = y'$, then it does indeed satisfy

$$\langle Tx, y\rangle = \langle x, T^* y\rangle,$$

for all $x, y \in H$.