MATH31001/41001/61001 Linear Analysis

Solution Sheet 5

1. This is not a linear functional, since $f(-x) = f(x) \neq -f(x)$ unless $x = 0$.

2. By definition,

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_2}.$$

To make $|f(x)|/\|x\|_2$ as large as possible, we can assume that $x_i = 0$ for $i \geq 3$, since this decreases $\|x\|_2$ but keeps $f(x)$ unchanged. Thus we need to maximize

$$\frac{|f(x_1, x_2, 0, \ldots)|}{\sqrt{x_1^2 + x_2^2}} = \frac{|x_1 - 3x_2|}{\sqrt{x_1^2 + x_2^2}}$$

over $(x_1, x_2) \neq (0, 0)$. We have

$$|f(x)|^2 = (x_1 - 3x_2)^2 = x_1^2 - 6x_1x_2 + 9x_2^2 \leq 10(x_1^2 + x_2^2),$$

whence

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_2} \leq \sqrt{10}.$$

Now spotting that the particular choices $x_1 = 1$ and $x_2 = -3$ gives $|x_1 - 3x_2| = 10$ and $\sqrt{x_1^2 + x_2^2} = \sqrt{10}$, we have

$$\|f\| \geq \frac{10}{\sqrt{10}} = \sqrt{10}.$$

Thus, $\|f\| = \sqrt{10}$.

(This is of course a special case of the result that $\|f_y\| = \|y\|$ in a Hilbert space; here $y = (1, -3, 0, 0, 0, \ldots)$ and $f(x) = f_y(x) = \langle x, y \rangle$.)

3. (a) That f_1 is linear follows from the basic rules of integration. To see that f_1 is bounded, note that

$$|f_1(\psi)| = \left| \int_0^1 x\psi(x)\, dx \right| \leq \int_0^1 x|\psi(x)|\, dx \leq \int_0^1 x\, dx\, \|\psi\|_\infty = \frac{1}{2}\|\psi\|_\infty$$

(we don't need to take $|x|$ because $x \geq 0$ on $[0, 1]$) and this also shows that $\|f\| \leq 1/2$.

To show that $\|f_1\| = 1/2$, take $\psi_1 = 1$. Then $\|\psi_1\|_\infty = 1$ and

$$f_1(\psi_1) = \int_0^1 x\, dx = \frac{1}{2}.$$

Thus $\|f_1\| \geq 1/2$ and so $\|f_1\| = 1/2$.

(b) That f_2 is linear follows from the basic rules of integration. To see that f_2 is bounded, note that

$$|f_2(\psi)| = \left|\int_0^1 x^2\psi(x)\ dx\right| \leq \int_0^1 x^2|\psi(x)|\ dx \leq \int_0^1 x^2\ dx\ \|\psi\|_\infty = \frac{1}{3}\|\psi\|_\infty$$

and this also shows that $\|f\| \leq 1/3$.

To show that $\|f_2\| = 1/3$, take $\psi_2 = 1$. Then $\|\psi_2\|_\infty = 1$ and

$$f_2(\psi_2) = \int_0^1 x^2\ dx = \frac{1}{3}.$$

Thus $\|f_2\| \geq 1/3$ and so $\|f_2\| = 1/3$.

(c) That f_3 is linear follows from the basic rules of integration. To see that f_3 is bounded, note that

$$\begin{aligned}|f_3(\psi)| &= \left|\int_0^1 \sin(2\pi x)\psi(x)\ dx\right| \leq \int_0^1 |\sin(2\pi x)|\ |\psi(x)|\ dx \\ &\leq \int_0^1 |\sin(2\pi x)|\ dx\ \|\psi\|_\infty = \frac{2}{\pi}\|\psi\|_\infty\end{aligned}$$

and this also shows that $\|f\| \leq 2/\pi$.

Showing that $\|f_3\| = 2/\pi$ is a bit more difficult. We would like to find $\psi_3 \in C([0,1],\mathbb{R})$ such that

$$\frac{|f_3(\psi_3)|}{\|\psi_3\|_\infty} := \frac{1}{\|\psi_3\|_\infty}\int_0^1 \sin(2\pi x)\psi_3(x)\ dx = \int_0^1 |\sin(2\pi x)|\ dx.$$

Clearly, $\psi_3 = 1$ doesn't do this. A function that would work is

$$\tau(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} \\ -1, & \frac{1}{2} < x \leq 1 \end{cases},$$

since $\sin(2\pi x)\tau(x) = |\sin(2\pi x)|$, but τ is *not* continuous. We will make the best of a bad job by trying to use continuous functions which approximate τ.

OK, here goes. For $n \geq 2$, define

$$\phi_n(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n\left(\frac{1}{2} - x\right), & \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2} + \frac{1}{n} \\ -1, & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}.$$

(Draw it!) Then, for each n, $\|\phi_n\|_\infty = 1$ and

$$\begin{aligned}
\frac{|f_3(\phi_n)|}{\|\phi_n\|_\infty} &= \left|\int_0^1 \sin(2\pi x)\phi_n(x)\ dx\right| \\
&= \left|\int_0^{1/2-1/n} |\sin(2\pi x)|\ dx + \int_{1/2-1/n}^{1/2+1/n} \sin(2\pi x)\phi_n(x)\ dx + \int_{1/2+1/n}^{1} |\sin(2\pi x)|\ dx\right| \\
&= \left|\int_0^1 |\sin(2\pi x)|\ dx - \int_{1/2-1/n}^{1/2+1/n} |\sin(2\pi x)|\ dx + \int_{1/2-1/n}^{1/2+1/n} \sin(2\pi x)\phi_n(x)\ dx\right| \\
&\geq \int_0^1 |\sin(2\pi x)|\ dx - \frac{2}{n} - \frac{2}{n} \\
&= \int_0^1 |\sin(2\pi x)|\ dx - \frac{4}{n}.
\end{aligned}$$

Now, given $\epsilon > 0$, we can choose n sufficiently large that $4/n < \epsilon$, and then

$$\frac{|f_3(\phi_n)|}{\|\phi_n\|_\infty} > \int_0^1 |\sin(2\pi x)|\ dx - \epsilon.$$

Thus,

$$\|f_3\| = \sup_{\psi\neq 0} \frac{|f_3(\psi)|}{\|\psi\|_\infty} \geq \int_0^1 |\sin(2\pi x)|\ dx.$$

Combining with the upper bound,

$$\|f_3\| = \int_0^1 |\sin(2\pi x)|\ dx = \frac{2}{\pi}.$$

4. It is easy to see that f is linear. To see that f_3 is bounded, note that

$$|f(\phi)| = |\phi(y)| \leq \|\phi\|_\infty,$$

which also shows that $\|f\| \leq 1$. If we take $\phi = 1$ then $\|\phi\|_\infty = 1$ and

$$|f(\phi)| = |f(1)| = 1.$$

Thus

$$\|f\| = \sup_{\phi\neq 0} \frac{|f(\phi)|}{\|\phi\|_\infty} \geq 1$$

and so $\|f\| = 1$.

5. Let us show that $\|\cdot\|'$ is a norm on X/W. The zero element in X/W is the coset $W = 0 + W$. Since $0 \in W$, tt is clear that

$$\|0 + W\|' = \inf\{\|w\| : w \in W\} = 0.$$

Next we need to show that this is the only coset with zero norm. Suppose that

$$\|x + W\|' = \inf\{\|x + w\| : w \in W\} = 0.$$

Then we can find a sequence $w_n \in W$ such that $\|x + w_n\| \leq 1/n$. Rewriting this as $\|w_n - (-x)\| \leq 1/n$, we see that $w_n \to -x$, as $n \to +\infty$. Since W is closed, we have $-x \in W$, so $x \in W$, i.e., $x + W = W = 0 + W$.

If λ is a scalar then $\lambda(x + W) = \lambda x + W$ and

$$\begin{aligned} \|\lambda x + W\|' &= \inf\{\|\lambda x + w\| : w \in W\} \\ &= |\lambda| \inf\left\{\left\|x + \frac{w}{\lambda}\right\| : w \in W\right\} \\ &= |\lambda| \|x + W\|'. \end{aligned}$$

Finally, we need to verify the triangle inequality: we need to show that

$$\|(x_1 + x_2) + W\|' \leq \|x_1 + W\|' + \|x_2 + W\|'.$$

For $\varepsilon > 0$, choose $w_1, w_2 \in W$ such that

$$\|x_1 + w_1\| \leq \|x_1 + W\|' + \varepsilon/2, \quad \text{and} \quad \|x_2 + w_2\| \leq \|x_2 + W\| + \varepsilon/2.$$

Then

$$\begin{aligned} \|(x_1 + x_2) + W\|' &\leq \|(x_1 + x_2) + (w_1 + w_2)\| \\ &\leq \|x_1 + w_1\| + \|x_2 + w_2\| \\ &\leq \|x_1 + W\|' + \|x_2 + W\|' + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives the result.