MATH31001/41001/61001 Linear Analysis

Solution Sheet 4

1. We will use the fact that the trace of a square matrix is the sum of its eigenvalues. Let us verify all the axioms of an inner product:

(1) We have $\langle A, A\rangle = \mathrm{trace}(AA^T)$; the matrix $B = AA^T$ is symmetric, and all its eigenvalues are nonnegative (exercise!). Hence $\langle A, A\rangle \geq 0$. It is obvious that $\langle A, A\rangle > 0$ unless $A = 0$, since a nonzero matrix has at least one nonzero eigenvalue.

(2) Since the trace of the transpose of a matrix is the same as that of the matrix itself, we have

$$\langle B, A\rangle = \mathrm{trace}(BA^T) = \mathrm{trace}((BA^T)^T) = \mathrm{trace}(AB^T) = \langle A, B\rangle.$$

(3) If the eigenvalues of A are $\{\lambda_1, \dots, \lambda_n\}$ and the eigenvalues of B are $\{\mu_1, \dots, \mu_n\}$, then the eigenvalues of $\alpha A + \beta B$ are $\{\alpha\lambda_1 + \beta\mu_1, \dots, \alpha\lambda_n + \beta\mu_n\}$, whence

$$\mathrm{trace}(\alpha A + \beta B) = \alpha \cdot \mathrm{trace}(A) + \beta \cdot \mathrm{trace}(B).$$

Hence $\langle \alpha A + \beta B, C\rangle = \alpha\langle A, C\rangle + \beta\langle B, C\rangle$.

2. We have

$$\langle x, y\rangle = 1/4 - 1/16 + 1/64 - 1/256 + \cdots = 1/5$$

and

$$\|x\| = \|y\| = 1/\sqrt{3}.$$

Hence the cosine of the angle in question equals $\langle x, y\rangle/(\|x\| \cdot \|y\|) = 3/5$ and the angle is $\cos^{-1}(3/5)$.

3. By definition $(L^\perp)^\perp$ is the set of vectors $x \in H$ such that $\langle x, y\rangle = 0$ for all $y \in L^\perp$. If $x \in L$ then

$$\langle x, y\rangle = \langle y, x\rangle = 0 \quad \text{for all } y \in L^\perp,$$

so that $x \in (L^\perp)^\perp$. Thus $L \subset (L^\perp)^\perp$.

From the question, $H = L \oplus L^\perp$. Suppose $x \in (L^\perp)^\perp \subset H$. Then we may write $x = x' + y$, for unique $x' \in L$ and $y \in L^\perp$. By the definition of $(L^\perp)^\perp$ and $L^\perp$,

$$0 = \langle x, y\rangle = \langle x', y\rangle + \langle y, y\rangle = \|y\|^2.$$

Thus, $y = 0$ and so $x = x' \in L$. Therefore $(L^\perp)^\perp \subset L$ and so $L = (L^\perp)^\perp$.

4. (a) L is linear: if $x, y \in L$, then $\alpha x + \beta y \in L$ as well, because

$$\alpha x_1 + \beta y_1 = \alpha \cdot 0 + \beta \cdot 0 = 0.$$

L is closed: let $x^{(n)} = (0, x_2^{(n)}, x_3^{(n)}, \dots) \in L$ and suppose $\|x^{(n)} - x\|_2 \to 0$, as $n \to +\infty$, for some $x = (x_1, x_2, \dots) \in \ell^2$. Then for any $\varepsilon > 0$ there exists n such that

$$|x_1|^2 + \sum_{i=2}^{\infty} |x_i^{(n)} - x_i|^2 < \varepsilon^2,$$

which gives $|x_1| < \varepsilon$. Since ε is arbitrary, $x_1 = 0$, i.e., $x \in L$.

(b) By definition,

$$L^\perp = \left\{ y = (y_1, y_2, \dots) \in \ell^2 : \sum_{i=2}^{\infty} x_i \overline{y}_i = 0 \ \forall (0, x_2, \dots) \in L \right\}.$$

For $i \geq 2$, the vector $x = e_i$ (1 in the ith place, 0 elsewhere) is in L. Thus, for $y \in L^\perp$, $y_i = \langle y, e_i \rangle = 0$, for all $i \geq 2$, whence $L^\perp \subset \{(y_1, 0, 0, \dots) : y_1 \in \mathbb{C}\}$. On the other hand, for $x = (0, x_2, \dots) \in L$, $\langle x, (y_1, 0, 0, \dots) \rangle = 0$, so $\{(y_1, 0, 0, \dots) : y_1 \in \mathbb{C}\} \subset L^\perp$. Therefore

$$L^\perp = \{(y_1, 0, 0, \dots) : y_1 \in \mathbb{C}\}.$$

(c) Every $x = (x_1, x_2, \dots) \in \ell^2$ can be expressed in a unique way in the form $x = y + z$, where $y = (0, x_2, x_3, \dots)$ and $z = (x_1, 0, 0, \dots)$. Hence $L \oplus L^\perp = H$.

5. The fact that ℓ^1 is linear follows from $x, y \in \ell^1 \implies \alpha x + \beta y \in \ell^1$ for any $\alpha, \beta \in \mathbb{C}$. Let us compute $L^\perp$:

$$L^\perp = \left\{ y = (y_1, y_2, \dots) \in \ell^2 : \sum_{i=1}^{\infty} x_i \overline{y}_i = 0 \ \ \forall x = (x_1, x_2, \dots) \in \ell^1 \right\}.$$

As in question 4, put $x = e_i \in \ell^1$, which implies $y_i = 0$ for all $i \geq 1$, i.e., $y = 0$. Hence $L^\perp = \{0\}$. Applying the formula $H = \overline{L} \oplus L^\perp$, we have $\ell^2 = \overline{\ell^1}$ (where the closure is taken with respect to $\|\cdot\|_2$), i.e., ℓ^1 is $\|\cdot\|_2$-dense in ℓ^2.

6. Since H is infinite-dimensional, there exists an infinite orthonormal set $D = \{e_n\}_{n=1}^{\infty}$ (i.e., $\langle e_n, e_m \rangle = \delta_{mn}$). Suppose S is compact; then there exists a subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ such that $\|e_{n_k} - x\| \to 0$, as $k \to +\infty$, for some $x \in H$. Notice that $\|e_{n_k}\| \to \|x\|$ as $k \to +\infty$, whence $\|x\| = 1$, i.e., $x \in S$.

We have $\|x - e_{n_k}\|^2 \to 0$, as $k \to +\infty$, so that

$$\begin{aligned} \|x - e_{n_k}\|^2 &= \langle x - e_{n_k}, x - e_{n_k} \rangle \\ &= \langle x, x \rangle - 2\langle x, e_{n_k} \rangle + \langle e_{n_k}, e_{n_k} \rangle \\ &= 2 - 2\langle x, e_{n_k} \rangle \to 0, \quad \text{as } k \to +\infty. \end{aligned}$$

For this to be true, we must have $\langle x, e_{n_k} \rangle \to 1$, as $k \to +\infty$, which contradicts Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_{n_k} \rangle|^2 \leq \|x\|^2 = 1,$$

as this requires $\langle x, e_{n_k} \rangle \to 0$, as $k \to +\infty$.

7. To simplify notation, we shall just give proofs for the real ℓ^p, ℓ^∞ spaces.
The space ℓ^p is separable for $1 \le p < \infty$: the set

$$C_N = \{(x_i)_{i=1}^\infty \mid x_i \in \mathbb{Q},\ x_i = 0 \text{ for } i \ge N\}$$

is countable and therefore so is

$$C = \bigcup_{N=0}^\infty C_N.$$

Suppose that $(y_i)_{i=1}^\infty \in \ell^p$ and choose $\varepsilon > 0$. By definition, there exists $N \ge 0$ such that $\left(\sum_{i=N}^\infty |y_i|^p\right)^{1/p} < \varepsilon$. Also, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $(x_i)_{i=1}^\infty \in C_N$ such that

$$|y_i - x_i| < \frac{\varepsilon}{2^i},\ i = 1, \dots, N-1.$$

Then

$$\|(y_i)_{i=1}^\infty - (x_i)_{i=1}^\infty\|_p = \left(\sum_{i=1}^{N-1} |y_i - x_i|^p + \sum_N^\infty |y_i|^p\right)^{1/p} < \varepsilon \left(\sum_{i=0}^{N-1} \frac{1}{2^i}\right)^{1/p} < 2^{1/p}\varepsilon.$$

Therefore C is dense in ℓ^p.
The space ℓ^∞ is not separable: Suppose that $E = \{(x_i^{(n)})_{i=1}^\infty\}_{n=1}^\infty$ is a countable subset of ℓ^∞. We shall show that this set cannot be dense by constructing an element $(y_i)_{i=1}^\infty \in \ell^\infty$ which is distance at least 1 from each element of E. We define $(y_i)_{i=1}^\infty$ by $y_i = x_i^{(i)} + 1$. Then, for each $n \ge 1$,

$$\|(y_i)_{i=1}^\infty - (x_i^{(n)})_{i=1}^\infty\|_\infty = \sup_{i \ge 1} |y_i - x_i^{(n)}| \ge |y_n - x_n^{(n)}| = 1.$$