MATH31001/41001/61001 Linear Analysis

Solution Sheet 3

1. First suppose that $1 \leq p < q$, where q is finite. To prove that $\ell^p \subset \ell^q$, we need to show that for any vector $(x_1, x_2, \ldots)$ such that $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, one also has $\sum_{i=1}^{\infty} |x_i|^q < +\infty$. So suppose that $\sum_{i=1}^{\infty} |x_i|^p < +\infty$; then we have $|x_i| \to 0$ as $i \to \infty$. Thus, in particular, there exists $C > 0$ such that $|x_i| \leq C$, for all $i = 1, 2, \ldots$. (Why?) Now,

$$|x_i|^q = |x_i|^{q-p}|x_i|^p \leq C^{q-p}|x_i|^p,$$

whence

$$\sum_{i=1}^{\infty} |x_i|^q \leq C^{q-p} \sum_{i=1}^{\infty} |x_i|^p < +\infty.$$

Now we show that these spaces are not equal. Take $x = (x_1, x_2, \ldots)$ given by $x_i = \frac{1}{i^{1/p}}$; then

$$\sum_{i=1}^{\infty} \left| \frac{1}{i^{1/p}} \right|^p = \sum_{i=1}^{\infty} \frac{1}{i} = +\infty$$

(the harmonic series!), whereas

$$\sum_{i=1}^{\infty} \left| \frac{1}{i^{1/p}} \right|^q = \sum_{i=1}^{\infty} \frac{1}{i^{q/p}} < +\infty,$$

because $q > p$ and the series $\sum_{i=1}^{\infty} \frac{1}{n^r}$ converges for $r > 1$. Hence $\ell^p \neq \ell^q$.

To complete the question, we shall show that $\ell^p \subset \ell^\infty$ but that $\ell^p \neq \ell^\infty$. If $(x_1, x_2, \ldots) \in \ell^p$ then $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, so that $|x_i| \to 0$ as $i \to \infty$. In particular $|x_i|$ is bounded and so $(x_1, x_2, \ldots) \in \ell^\infty$. Thus $\ell^p \subset \ell^\infty$. Now consider the vector $(1, 1, 1, \ldots)$ (all entries 1). Clearly, $(1, 1, 1, \ldots) \in \ell^\infty$ but

$$|1|^p + |1|^p + |1|^p + \cdots = 1 + 1 + 1 + \cdots = +\infty,$$

so $(1, 1, 1, \ldots) \notin \ell^p$. Thus $\ell^p \neq \ell^\infty$.

2. Put

$$x_i^{(n)} = \begin{cases} \frac{1}{i}, & 1 \leq i \leq n, \\ 0, & i > n \end{cases}$$

Then

$$\|x^{(n)}\|_1 = \sum_{i=1}^{n} \frac{1}{i} \to +\infty \text{ as } n \to +\infty,$$

whilst

$$\|x^{(n)}\|_2 = \left(\sum_{i=1}^{n} \frac{1}{i^2} \right)^{1/2} \to \left(\frac{\pi^2}{6} \right)^{1/2} < +\infty \text{ as } n \to +\infty.$$

In particular, $\lim_{n\to+\infty} \|x^{(n)}\|_1/\|x^{(n)}\|_2 = +\infty$. Thus these norms are not equivalent.

3. Use the sequence $x^{(n)}$ from Problem 2. First we see that is a Cauchy sequence with respect to $\|\cdot\|_2$: if $n > m$ then

$$\|x^{(n)} - x^{(m)}\|_2 = \left(\sum_{i=m+1}^{n} \frac{1}{i^2} \right)^{1/2} \to 0, \quad \text{as } n, m \to +\infty.$$

Also, if $x = (x_1, x_2, \ldots)$ is given by $x_i = 1/i$ then $x \in \ell^2$ and

$$\|x^{(n)} - x\|_2 = \left(\sum_{i=n+1}^{\infty} \frac{1}{i^2} \right)^{1/2} \to 0, \quad \text{as } n \to +\infty,$$

i.e., $x^{(n)}$ converges to x in $(\ell^2, \|\cdot\|_2)$ and $x \notin \ell^1$. However, limits are unique, so $x^{(n)}$ cannot converge with respect to $\|\cdot\|_2$ to any other vector in $\ell^2 \supset \ell^1$. In particular, the Cauchy sequence $x^{(n)}$ does not converge in $(\ell^1, \|\cdot\|_2)$, so $(\ell^1, \|\cdot\|_2)$ is not a Banach space.

4. First check that $\|\cdot\|_1$ is indeed a norm. It is clear that $\|\lambda f\|_1 = |\lambda|\|f\|_1$ and that $\|0\|_1 = 0$. If $f \neq 0$ then there exists $x_0 \in (0,1)$ such that $|f(x_0)| = a > 0$. By continuity, there exists $\delta > 0$ such that $|f(x)| \geq a/2$ for $x \in (x_0 - \delta, x_0 + \delta)$. Thus (since $|f| \geq 0$),

$$\|f\|_1 = \int_0^1 |f(x)|dx \geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|dx \geq \delta a > 0,$$

so $\|f\|_1 = 0$ if and only if $f = 0$. Finally,

$$\|f + g\|_1 = \int_0^1 |f(x) + g(x)|dx \leq \int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx = \|f\|_1 + \|g\|_1.$$

To see that $(V, \|\cdot\|_1)$ is not a Banach space, we will find a Cauchy sequence which does not converge in V with respect to the norm $\|\cdot\|_1$. Consider the sequence:

$$f_n(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} - \frac{1}{n} \\ nx + 1 - n/2, & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}.$$

(To aid understanding, draw this function!) To check that this is a Cauchy sequence, notice that, for any $m > n$, $f_n - f_m \geq 0$, so that

$$\begin{aligned}\|f_m - f_n\|_1 &= \int_0^1 |f_n(x) - f_m(x)| dx = \int_0^1 f_n(x) - f_m(x)\ dx \\ &= \int_0^{1/2} f_n(x) - f_m(x)\ dx \leq \int_0^{1/2} f_n(x)\ dx \\ &= \int_{1/2-1/n}^{1/2} (nx + 1 - n/2)\ dx \\ &= \frac{1}{2n} \to 0, \quad \text{as } n, m \to +\infty.\end{aligned}$$

(The simple way to evaluate the integral is to use the $\frac{1}{2} \times \text{(base)} \times \text{(height)}$ formula for the area of a triangle.) Now, the pointwise limit of f_n is

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

which is clearly not continuous, and a simple calculation shows that $\|f_n - f\|_1 \to 0$, as $n \to +\infty$. However, we are not finished because it might be that there is some continuous function g for which $\|f_n - g\|_1 \to 0$, as $n \to +\infty$. We shall rule this out by arguing by contradiction. Suppose that g is continuous and that $\lim_{n\to+\infty} \|f_n - g\|_1 = 0$. For each $n \geq 1$, we have

$$\|g - f\|_1 \leq \|g - f_n\|_1 + \|f_n - f\|_1,$$

so, letting $n \to +\infty$, we get $\|g - f\|_1 = 0$, i.e.,

$$0 = \int_0^1 |g(x) - f(x)| dx = \int_0^{1/2} |g(x)| dx + \int_{1/2}^1 |g(x) - 1| dx.$$

For this to hold, both integrals must be zero:

$$\int_0^{1/2} |g(x)| dx = 0 \quad \text{and} \quad \int_{1/2}^1 |g(x) - 1| dx = 0.$$

Since g and $g-1$ are assumed continuous, we may argue as in the first part of the question to conclude that $g(x) = 0$ for $x \in [0, 1/2]$ and $g(x) = 1$ for $x \in [1/2, 1]$. Thius contradicts g being continuous (or even well-defined) on $[0, 1]$. To summarize, we have constructed a sequence of functions in $C([0,1], \mathbb{R})$ which is Cauchy sequence with respect to the norm $\|\cdot\|_1$ but which does not converge to a continuous function with respect to this norm. Thus, $(V, \|\cdot\|_1)$ is not a Banach space.

The following instructive example shows that, to prove $(V, \|\cdot\|_1)$ is not a Banach space, it is not sufficient to find a Cauchy sequence which converges pointwise to a discontinuous function. Consider the sequence:

$$h_n(x) = \begin{cases} 1 - nx, & 0 \leq x \leq 1/n \\ 0, & 1/n < x \leq 1 \end{cases}.$$

To check that this is a Cauchy sequence, notice that for any $m > n$,

$$\begin{aligned} \|h_n - h_m\|_1 &= \int_0^{1/n} (1 - nx)\; dx - \int_0^{1/m} (1 - mx)\; dx \\ &\leq \int_0^{1/n} (1 - nx)\; dx = \frac{1}{2n} \to 0, \quad \text{as } n, m \to +\infty. \end{aligned}$$

The pointwise limit of this sequence is

$$h(x) = \begin{cases} 1, & x = 0 \\ 0, & 0 < x \leq 1, \end{cases}$$

which is not continuous. However,

$$\|h_n - 0\|_1 = \|h_n\|_1 = \int_0^{1/n} (1 - nx)\; dx = \frac{1}{2n} \to 0, \quad n \to +\infty,$$

so h_n converges to the continuous function 0 with respect to the norm $\|\cdot\|_1$.

5. $\|\cdot\|_1$ is not a norm on $R([0,1],\mathbb{R})$ because it contains functions $f \neq 0$ such that $\|f\|_1 = 0$. For example, using h from the last question, define

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & 0 < x \leq 1, \end{cases}.$$

Clearly, $f \neq 0$ but

$$\|f\|_1 = \int_0^1 |f(x)| dx = 0.$$

Additional note (not examinable): From questions 4. and 5. it might look like $\|\cdot\|_1$ is a bit of a loser but this is a false impression – given the correct vector space $\|\cdot\|_1$ gives a Banach space. For those who have studied Lebesgue integration, I can explain this. Let λ denote Lebesgue measure and let $\mathcal{L}^1([0,1],\lambda)$ denote the vector space of all integrable functions $f : [0,1] \to \mathbb{R}^*$, i.e., all Lebesgue measurable functions $f : [0,1] \to \mathbb{R}^*$ for which

$$\int_{[0,1]} |f| d\lambda < +\infty.$$

Define an equivalence relation on $\mathcal{L}^1([0,1],\lambda)$ by $f \sim g$ if $f = g$ λ-almost everywhere and let $L^1([0,1],\lambda)$ denote the set of equivalence classes. If $f = g$ λ-almost everywhere then $|f| = |g|$ λ-almost everywhere and this implies that

$$\int_{[0,1]} |f| d\lambda = \int_{[0,1]} |g| d\lambda.$$

Thus

$$\|f\|_1 = \int_{[0,1]} |f| d\lambda$$

is well-defined on $L^1([0,1],\lambda)$ and, in fact, makes it into a Banach space.

6*. If V is a Banach space then, by definition, V is complete, so S is a closed subset of a complete set and is hence complete.

If S is complete, consider a Cauchy sequence x_n, $n \geq 1$, in V. To show that V is complete (and hence a Banach space), we need to show that x_n converges. If x_n converges to 0 we are done. If x_n does not converge to 0 then neither does any subsequence, so there exists $N \geq 1$ and $\varepsilon > 0$ such that $\|x_n\| \geq \varepsilon$ for all $n \geq N$. Now consider the sequence $x_n/\|x_n\|$, $n \geq 1$ in S. Then, for $n, m \geq N$,

$$\begin{aligned} \left\| \frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|} \right\| &\leq \frac{\|x_n - x_m\|}{\min\{\|x_n\|, \|x_m\|\}} + \frac{|\|x_n\| - \|x_n\||}{\min\{\|x_n\|, \|x_m\|\}} \\ &\leq \frac{2\|x_n - x_m\|}{\varepsilon} \to 0, \quad \text{as } n, m \to +\infty. \end{aligned}$$

(More detail on the derivation of this inequality over the page.) Therefore, $x_n/\|x_n\|$ is a Cauchy sequence in S and so it converges to $y \in S$, say. Since

$$|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\|,$$

$\|x_n\|$ is a Cauchy sequence in $\mathbb{R}$ and so it converges to some $l \in \mathbb{R}$. Combining these two facts, we have

$$\begin{aligned} \|x_n - ly\| &\leq \|x_n - \|x_n\| \cdot y\| + \|\|x_n\| \cdot y - ly\| \\ &= \|x_n\| \left\| \frac{x_n}{\|x_n\|} - y \right\| + \|y\|(\|x_n\| - l) \\ &\to l \cdot 0 + \|y\| \cdot 0 = 0, \quad \text{as } n \to +\infty. \end{aligned}$$

Hence $x_n \to ly$, as $n \to +\infty$, and V is complete.

Here is some extra detail on deriving the inequality indicated above. To simplify notation, suppose that $\|x_n\|$ is the smaller of the norms (i.e. $\|x_n\| = \min\{\|x_n\|, \|x_m\|\}$).

$$\begin{aligned}
\frac{1}{\|x_n\|}\left\|x_n - \frac{\|x_n\|x_m}{\|x_m\|}\right\| &= \frac{1}{\|x_n\|}\left\|x_n - x_m + x_m - \frac{\|x_n\|x_m}{\|x_m\|}\right\| \\
&\leq \frac{1}{\|x_n\|}\|x_n - x_m\| + \frac{1}{\|x_n\|}\left\|x_m - \frac{\|x_n\|x_m}{\|x_m\|}\right\| \\
&= \frac{1}{\|x_n\|}\|x_n - x_m\| + \frac{1}{\|x_n\|}\left\|\frac{\|x_m\|x_m}{\|x_m\|} - \frac{\|x_n\|x_m}{\|x_m\|}\right\| \\
&= \frac{1}{\|x_n\|}\|x_n - x_m\| + \frac{1}{\|x_n\|}\left\|(\|x_m\| - \|x_n\|)\frac{x_m}{\|x_m\|}\right\| \\
&= \frac{1}{\|x_n\|}\|x_n - x_m\| + \frac{1}{\|x_n\|}|\|x_m\| - \|x_n\||
\end{aligned}$$

(since $\|x_m/\|x_m\|\| = 1$).