MATH31001/41001/61001 Linear Analysis

Solution Sheet 2

1. Consider the set $\mathcal{R}$ of the polynomials with rational coefficients:

$$\mathcal{R} = \left\{ \sum_{k=0}^{n} a_k x^k \mid a_k \in \mathbb{Q},\ 0 \le k \le n,\ n \ge 0 \right\}.$$

The set $\mathcal{R}$ is clearly countable (as $\mathbb{Q}$ is countable) and we claim it to be dense in $C[0,1]$. Let us prove it.

Fix $\varepsilon > 0$ and a function $f \in C[0,1]$. By the Weierstrass Theorem, there exists a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ with **real** coefficients a_k such that $\|f - p\|_\infty < \varepsilon/2$.

Now, for any k, we choose a rational number r_k in such a way that $|a_k - r_k| < \frac{\varepsilon}{2(n+1)}$. Then

$$\left\| \sum_{k=0}^{n} (a_k - r_k) x^k \right\|_\infty \le \sum_{k=0}^{n} |a_k - r_k| < \varepsilon/2.$$

Put $R(x) = \sum_{k=0}^{n} r_k x^k \in \mathcal{R}$. Hence

$$\|f - R\|_\infty \le \|f - p\|_\infty + \|p - R\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Remark. Notice that $\mathcal{R}$ is **not** an algebra over $\mathbb{R}$ so we cannot apply the Stone-Weierstrass Theorem directly.

2. If $(a_i)_{i=0}^{\infty}$ and $(b_i)_{i=0}^{\infty}$ both in ℓ^1, then so is $(\alpha a_i + \beta b_i)_{i=0}^{\infty}$ for any real α and β, since

$$\sum_{i=0}^{\infty} |\alpha a_i + \beta b_i| \le |\alpha| \sum_{i=0}^{\infty} |a_i| + |\beta| \sum_{i=0}^{\infty} |b_i| < \infty.$$

For ℓ^∞ use the fact that

$$\sup_i |\alpha a_i + \beta b_i| \le |\alpha| \sup_i |a_i| + |\beta| \sup_i |b_i|.$$

3. It is clear that $c_0 \subset l^\infty$ is a vector space. Suppose that $\{(x_i^{(n)})_{i=0}^{\infty}\}_{n=1}^{\infty}$ is a sequence in c_0 and that $\lim_{n\to+\infty} ||(x_i^{(n)})_{i=0}^{\infty} - (x_i)_{i=0}^{\infty}||_\infty = 0$, for some $(x_i)_{i=0}^{\infty} \in l^\infty$. Then, given $\varepsilon > 0$, there exists $n = n(\varepsilon) \ge 1$ such that, for all i,

$$|x_i^{(m)} - x_i| \le \|(x_i^{(m)})_{i=0}^{\infty} - (x_i)_{i=0}^{\infty}\|_\infty < \varepsilon/2, \quad \forall\ m \ge n.$$

Furthermore, from the definition of c_0, for each n, there exists $N(n) \ge 1$ such that

$$|x_i^{(n)}| < \varepsilon/2, \quad \forall\ i \ge N(n).$$

Thus, for $i \ge N(n)$,

$$|x_i| \le |x_i^{(n)} - x_i| + |x_i^{(n)}| < \varepsilon,$$

so $(x_i)_{i=0}^{\infty} \in c_0$ and c_0 is closed.

4. No, because if f is increasing, $-f$ is decreasing.

5. No. Take $f(x) = x$ and $g(x) = x^2$, both increasing. Then $f(x) - g(x) = x - x^2$, which is not monotonic on $[0, 1]$, as it has a maximum at $1/2$.