MATH31001/41001/61001 Linear Analysis

Solution Sheet 1

1. It is obvious that $d(x,y) = \|x-y\| \geq 0$ and $= 0$ if and only if $x = y$. Also, clearly, $d(x,y) = d(y,x)$. Let us check the triangle inequality:

$$d(x,y) + d(y,z) = \|x-y\| + \|y-z\| \geq \|x-y+y-z\| = \|x-z\| = d(x,z).$$

2. (a) Here

$$B_n(x;x) = \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} = x$$

(see Lemma 1.2).

(b) Here we have

$$\begin{aligned}
B_n(x^2;x) &= \sum_{k=0}^{n} \frac{k^2}{n^2}\binom{n}{k} x^k(1-x)^{n-k} \\
&= \frac{1}{n^2}\sum_{k=0}^{n} k^2 \binom{n}{k} x^k(1-x)^{n-k} \\
&= \frac{1}{n^2}\sum_{k=0}^{n} \big((k-nx)^2 + 2knx - n^2x^2\big)\binom{n}{k} x^k(1-x)^{n-k} \\
&= \frac{1}{n^2}(nx(1-x) + 2n^2x^2 - n^2x^2) \\
&= x^2 + \frac{x(1-x)}{n}
\end{aligned}$$

(also using Lemma 1.2). Hence $|B_n(x^2;x) - x^2| \leq 1/4n \to 0$, which shows that $B_n(x^2;x)$ converges uniformly to x^2.

(c) We shall use the following notation: for sequences a_n, b_n, we shall write $a_n = O(b_n)$ if there exists a constant $C \geq 0$ such that $|a_n| \leq C b_n$. The constant C is called the *implied constant.*

In this case,

$$\begin{aligned}
B_n(e^x;x) &= \sum_{k=0}^{n} e^{k/n}\binom{n}{k} x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (xe^{1/n})^k (1-x)^{n-k} \\
&= (xe^{1/n} + 1 - x)^n
\end{aligned}$$

(by the binomial formula). Notice that

$$e^{1/n} - 1 = \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

This gives us (again using the binomial formula)

$$\begin{aligned} B_n(e^x; x) &= (1 + x(e^{1/n} - 1))^n \\ &= \left(1 + \frac{x}{n} + O\left(\frac{x}{n^2}\right)\right)^n \\ &= \left(1 + \frac{x}{n}\right)^n + O\left(\frac{1}{n}\right), \end{aligned}$$

where one can check that the implied constant is independent of $x \in [0, 1]$.

One also knowns that

$$\ln\left(1 + \frac{x}{n}\right) = \frac{x}{n} + O\left(\frac{1}{n^2}\right),$$

where the implied constant in $O(1/n^2)$ does not depend on x, since $0 \leq x \leq 1$. Hence

$$\ln\left(1 + \frac{x}{n}\right)^n = x + O\left(\frac{1}{n}\right),$$

and

$$\left(1 + \frac{x}{n}\right)^n = e^{x+O(1/n)},$$

whence

$$\left(1 + \frac{x}{n}\right)^n - e^x = O\left(\frac{1}{n}\right),$$

with all the implied constants independent of $x \in [0, 1]$. Combining this with the the estimate

$$B_n(e^x; x) - \left(1 + \frac{x}{n}\right)^n = O\left(\frac{1}{n}\right)$$

shows that $B_n(e^x; x)$ converges uniformly to e^x, as $n \to +\infty$.

3. Since $[a, b] \times [c, d]$ is compact we want to apply the Stone-Weierstrass Theorem to the algebra

$$\mathcal{A} = \{p(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j\},$$

i.e., the set of polynomials on $[a, b] \times [c, d]$. This contains the non-zero constant function 1. To see that $\mathcal{A}$ separates points, we argue as follows. Suppose that $(x, y) \neq (x', y')$. If $x \neq x'$, take $p(x, y) = x$, so that $p(x', y') = x' \neq p(x, y)$. If $x = x'$ then $y \neq y'$ and we take $p(x, y) = y$, so that $p(x', y') = y' \neq p(x, y)$. In either case, we have found an element of $\mathcal{A}$ which distinguishes between (x, y) and (x', y'). Thus $\mathcal{A}$ satisfies the hypotheses of the Stone-Weierstrass Theorem and is hence uniformly dense in $C([a, b] \times [c, d], \mathbb{R})$, as required.

4. By the Stone-Weierstrass Theorem, given $\varepsilon > 0$, we can find a polynomial $p(x)$ so that $\|p - f\|_\infty < \varepsilon$. By our hypothesis, $M_n = 0$, so if $p(x) = \sum_{n=0}^{N} a_n x^n$ then $\int_0^1 f(x)p(x)dx = \sum_{n=0}^{N} a_n M_n = 0$. We now have that

$$\int_0^1 |f(x)|^2 dx = \int_0^1 (f(x))^2 dx = \int_0^1 f(x)(f(x) - p(x)) \leq \|f\|_\infty \cdot \varepsilon.$$

This is true for all $\varepsilon > 0$ so $\int_0^1 |f(x)|^2 dx = 0$. Since f is continuous, we conclude that $f = 0$.

5. Let $\mathcal{A}$ denote the set of all functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$, where $f_i \in C(X, \mathbb{R})$ and $g_i \in C(Y, \mathbb{R})$. We have:

(1) $\mathcal{A}$ is clearly an algebra;
(2) $1 = 1(x, y) = 1(x)1(y) \in \mathcal{A}$;
(3) $\mathcal{A}$ separates points: if $(x, y) \neq (x', y')$ then $x \neq x'$ or $y \neq y'$. In the former case, choose $f \in C(X)$ with $f(x) \neq f(x')$, then $p(x, y) = f(x) \times 1$ separates (x, y) and (x', y'). If $y \neq y'$, the argument is similar.

Thus, by the Stone-Weierstrass Theorem, $\mathcal{A}$ is uniformly dense in $C(X \times Y, \mathbb{R})$.