

MATH31001/41001/61001: LINEAR ANALYSIS

EXAMPLE SHEET 7

1. Let H be a Hilbert space and let $S, T : H \rightarrow H$ be bounded linear operators. Show that

- (i) $(ST)^* = T^*S^*$;
- (ii) $\|T^*T\| = \|TT^*\| = \|T\|^2$.

2. Let $H = \ell^2$ and, for $n \geq 1$, define a sequence of linear operators $T_n : \ell^2 \rightarrow \ell^2$ by

$$T_n(x_1, x_2, x_3, \dots) = (x_n, 0, 0, \dots).$$

Show that $\|T_n\| = 1$, for all $n \geq 1$.

Show that

- (i) T_1 has 0 and 1 as eigenvalues and no other eigenvalues;
- (i) for $n \geq 2$, T_n has 0 as an eigenvalue and no other eigenvalues.

3. Let $(a_i)_{i=1}^{\infty} \in \ell^{\infty}$ and define $T : \ell^2 \rightarrow \ell^2$ by $T((x_i)_{i=1}^{\infty}) = (a_i x_i)_{i=1}^{\infty}$.

- (i) Show that, for each $i \geq 0$, $a_i \in \text{spec}(T)$.
- (ii) Show that if $\lambda \notin \overline{\{a_i : i \geq 0\}}$, then $\lambda \notin \text{spec}(T)$.

Deduce that $\text{spec}(T) = \overline{\{a_i : i \geq 0\}}$.

4. Let $T : \ell^2 \rightarrow \ell^2$ be defined by

$$T((x_i)_{i=1}^{\infty}) = \left(\frac{x_{i+1}}{i} \right)_{i=1}^{\infty}.$$

Show that $\|T^n\| = 1/n!$ and conclude that $\text{spec}(T) = \{0\}$.

(Hint: you may use Stirling's formula: $n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}$ as $n \rightarrow +\infty$, where \sim means the ratio of the two sides tends to 1.)

5*. Let X be a Banach space and let $T \in B(X)$. Recall (from Example Sheet 6) that $T^* : X^* \rightarrow X^*$ is defined by $(T^*f)(x) = f(Tx)$.

- (i) Prove that $\|T^*\| \leq \|T\|$.
- (ii) [This requires Proposition E.4 in the 41001/61001 extra reading.] Show that $\|T^*\| \geq \|T\|$.

(THE STARRED EXERCISES ARE HARDER)