MATH31001/41001/61001: LINEAR ANALYSIS

Example Sheet 6

1. Let $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ be defined as follows:

$$(Tf)(x) = f(x^2).$$

Show that T is a bounded linear operator on $C([0,1],\mathbb{R})$ (with the norm $\|\cdot\|_\infty$) and compute its norm $\|T\|$.

2. Let $f \in C([0,1],\mathbb{R})$ and define an operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$T(g) = fg.$$

Show that T is a bounded linear operator on $C([0,1],\mathbb{R})$ (with the norm $\|\cdot\|_\infty$) and that $\|T\| = \|f\|_\infty$.

3. Let V be a normed vector space and let V' be a Banach space. Prove that $B(V, V')$ is a Banach space.

(Hint: follow the proof of a similar result for the dual space V^*.)

4. Let $H = \ell^2$ and let $a = (a_1, a_2, a_3, \dots) \in \ell^\infty$. Define

$$T_a(x_1, x_2, x_3, \dots) = (a_1x_1, a_2x_2, a_3x_3, \dots).$$

Show that $T_a : \ell^2 \to \ell^2$ is a bounded linear operator. Show that T_a is self-adjoint if and only if a is a real vector (i.e. $a_i \in \mathbb{R}$, for all $i \geq 1$).

5*. Given a Banach space V, and $T \in B(V)$, we define the *adjoint operator* $T^* : V^* \to V^*$ by

$$(T^*f)(x) = f(Tx), \quad \text{where } f \in V^*, x \in V.$$

Explain how this definition agrees with the one given in the lectures when $V = H$ is a Hilbert space.

(Hint: use the Riesz Representation Theorem to identify H and H^*.)

(THE STARRED EXERCISES ARE HARDER)