MATH31001/41001/61001: LINEAR ANALYSIS

Example Sheet 5

1. Let $V = \ell^1$ and put $f(x) = \|x\|_1$. Is f a linear functional on V? Justify your answer.

2. Let $V = \ell^2$ and define $f \in V^*$ by $f(x) = x_1 - 3x_2$ for $x = (x_1, x_2, \dots) \in V$. Compute $\|f\|$.

3. Consider $V = C([0,1], \mathbb{R})$ with the norm $\|\cdot\|_\infty$. Define $f_1, f_2, f_3 : V \to \mathbb{R}$ by

$$f_1(\psi) = \int_0^1 x\psi(x)\ dx, \quad f_2(\psi) = \int_0^1 x^2\psi(x)\ dx, \quad f_3(\psi) = \int_0^1 \sin(2\pi x)\psi(x)\ dx.$$

Show that

(a) $f_1 \in V^*$ and that

$$\|f_1\| = \int_0^1 x\ dx = \frac{1}{2};$$

(b) $f_2 \in V^*$ and that

$$\|f_2\| = \int_0^1 x^2\ dx = \frac{1}{3};$$

(c)***** $f_3 \in V^*$ and that

$$\|f_3\| = \int_0^1 |\sin(2\pi x)|\ dx = \frac{2}{\pi}.$$

4. Consider $V = C([0,1], \mathbb{R})$ with the norm $\|\cdot\|_\infty$ and choose $y \in [0,1]$. Define $f : V \to \mathbb{R}$ by $f(\phi) = \phi(y)$. Show that $f \in V^*$ and calculate $\|f\|$.

5*. Let $(V, \|\cdot\|)$ be a normed vector space and let $W \subset V$ be a closed linear subspace. For $x \in V$, write $x + W$ for the coset $x + W = \{x + w : w \in W\}$.

Show that the quotient space defined by

$$V/W = \{x + W : v \in X\}$$

is a normed vector space with the norm $\|x + W\|' = \inf\{\|x + w\| : w \in W\}$.

(THE STARRED EXERCISES ARE HARDER)