MATH31001/41001/61001: LINEAR ANALYSIS

Example Sheet 4

1. Let X denote the set of $n \times n$ real matrices. Put for any two matrices $A, B \in X$,

$$\langle A, B \rangle = \text{trace}(AB^T),$$

where B^T is the transpose of B. Show that $\langle \cdot, \cdot \rangle$ is indeed an inner product on X. *(Hint: use the fact that the trace of a square matrix is the sum of its eigenvalues.)*

2. Let $H = \ell^2$. As in finite dimensions, we may define the angle between two vectors x and y to be

$$\cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2}\right).$$

Compute the angle between $x = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$ and $y = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right)$ in H.

3. Let H be a Hilbert space and let L be a closed linear subspace. Show that $L \subset (L^\perp)^\perp$. Since L is closed, we have $H = L \oplus L^\perp$ (rather than just $H = \overline{L} \oplus L^\perp$). Deduce that $L = (L^\perp)^\perp$.

4. Put $H = \ell^2$ and $L = \{(x_1, x_2, \dots) \in H : x_1 = 0\}$.
- (a) Show that L is a closed linear subspace of H;
- (b) Compute $L^\perp$;
- (c) Verify that, in this example, $H = L \oplus L^\perp$.

5. Let $H = \ell^2$ and $L = \ell^1$. Verify that L is a linear subspace of H and compute $L^\perp$. What does the formula $H = \overline{L} \oplus L^\perp$ give us in this case?

6*. Let H be an infinite-dimensional Hilbert space over $\mathbb{R}$. Use Bessel's inequality to show that the unit sphere $S = \{x \in H : \|x\| = 1\}$ is **not** compact in H (albeit clearly closed and bounded!). *(Hint: if S were compact, the orthonormal set $\{e_n\}_{n=1}^\infty$ would have an accumulation point. Prove by contradiction that this is impossible.)*

7*. Show that any space ℓ^p is separable for $1 \le p < \infty$, whereas ℓ^∞ is not separable. *(Hint: for ℓ^∞, adapt the proof that $\mathbb{R}$ is uncountable.)*

(THE STARRED EXERCISES ARE HARDER)

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