MATH31001/41001/61001: LINEAR ANALYSIS

Example Sheet 3

1. Let $1 \leq p < q \leq \infty$. Show that $\ell^p \subset \ell^q$ but that $\ell^p \neq \ell^q$.

2. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent on the space ℓ^1. *(Hint: find a sequence of elements $x^{(n)}$ in ℓ^1 such that $\|x^{(n)}\|_1/\|x^{(n)}\|_2 \to +\infty$, as $n \to +\infty$.)*

3. Show that $(\ell^1, \|\cdot\|_2)$ is not a Banach space.

4*. Let V be the vector space of continuous functions $C([0,1],\mathbb{R})$ and define

$$\|f\|_1 = \int_0^1 |f(x)|dx.$$

Show that $\|\cdot\|_1$ is a norm on V but that $(V, \|\cdot\|_1)$ is *not* a Banach space.

5. Let $R([0,1],\mathbb{R})$ denote the space of Riemann integrable functions of $[0,1]$. Show that $\|\cdot\|_1$ is not a norm on $R([0,1],\mathbb{R})$.

6*. Let V be a normed linear space with norm $\|\cdot\|$. Prove that V is a Banach space if and only if the unit sphere

$$S = \{x \in V : \|x\| = 1\}$$

is complete for the induced metric (i.e., if $(x_n)_{n=0}^{\infty} \subset S$ satisfy $\|x_n - x_m\| \to 0$, as $n, m \to +\infty$, then there exists $x \in S$ with $\lim_{n\to+\infty} x_n = x$).

(THE STARRED EXERCISES ARE HARDER)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX