

MATH31001/41001/61001: LINEAR ANALYSIS

EXAMPLE SHEET 1

1. Let $(X, \|\cdot\|)$ be a normed space. Verify that the function $d(x, y) = \|x - y\|$ is a metric.
2. Compute the n th Bernstein polynomial $B_n(f; x)$ for the following functions:
 - (a) $f(x) = x$;
 - (b) $f(x) = x^2$.
 - (c)* $f(x) = e^x$;

Show directly that in all three cases $B_n(f; x) \rightarrow f(x)$ uniformly as $n \rightarrow +\infty$ on $[0, 1]$. (That is, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x) - B_n(f; x)| < \varepsilon$ for all $n \geq N$ and all $x \in [0, 1]$.)

3. Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a continuous function on the square $[a, b] \times [c, d]$. Use the Stone-Weierstrass Theorem to show that f can be approximated by a polynomial in the two variables x and y , i.e., a function of the form

$$p(x, y) = \sum_{i=0}^n \sum_{j=0}^m a_{ij} x^i y^j.$$

- 4.* Let $f \in C[0, 1]$. The n -th moment of f is defined to be $M_n = \int_0^1 f(x)x^n dx$, $n \geq 0$. Show that if $M_n = 0$, for all $n \geq 0$, then $f = 0$.

(Hint: Approximate f by polynomials and deduce that $\int_0^1 |f(x)|^2 dx = 0$.)

- 5.* Let X and Y be compact metric spaces. Let $X \times Y = \{(x, y) : x \in X, y \in Y\}$ be the cartesian product. Show that any $f \in C(X \times Y, \mathbb{R})$ can be uniformly approximated by functions of the form $\sum_{i=1}^n f_i(x)g_i(y)$, where $f_i \in C(X, \mathbb{R})$ and $g_i \in C(Y, \mathbb{R})$.

(A slight generalization of Problem 3.)

(THE STARRED EXERCISES ARE HARDER)