MATH31001/MATH41001/MATH61001: LINEAR ANALYSIS

PROOF OF PROPOSITION 3.2

Proposition 3.2. *If* $(V, \|\cdot\|)$ *is a normed vector space then* $(V^*, \|\cdot\|)$ *is a Banach space.*

Proof.
V^* is a vector space: Suppose that $f, g \in V^*$ and that λ is a scalar. Then

$$(\lambda f)(x) = \lambda f(x)$$

and

$$(f+g)(x) = f(x) + g(x)$$

and these are clearly continuous.
$\|\cdot\|$ is a norm:

(1) Clearly $\|f\| \geq 0$ and

$$\|f\| = 0 \quad \iff \quad \sup_{\|x\|=1} |f(x)| = 0 \quad \iff \quad |f(x)| = 0, \text{ for all } x \text{ with } \|x\| = 1$$

and the latter identity is equivalent to $f = 0$.

(2)

$$\|\lambda f\| = \sup_{\|x\|=1} |\lambda||f(x)| = |\lambda| \sup_{\|x\|=1} |f(x)| = |\lambda|\|f\|.$$

(3) For $\|x\| = 1$,

$$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq \sup_{\|x\|=1} |f(x)| + \sup_{\|x\|=1} |g(x)| = \|f\| + \|g\|. \end{aligned}$$

Taking the supremum, we get

$$\|f+g\| = \sup_{\|x\|=1} |f(x) + g(x)| \leq \|f\| + \|g\|.$$

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V^* is a Banach space: Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in V^*. Fixing $x \in V$,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|\|x\|,$$

so $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) and we may write $f(x) = \lim_{n\to+\infty} f_n(x)$. We need to show that this f is an element of V^*.

First we check that f is linear:

$$f(\lambda x + \mu y) = \lim_{n\to+\infty} f_n(\lambda x + \mu y) = \lim_{n\to+\infty} (\lambda f_n(x) + \mu f_n(y)) = \lambda f(x) + \mu f(y).$$

Next, we check that f is bounded (remember this is equivalent to continuous). Since $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence, we may choose $N \ge 1$ so that $n, m \ge N$ implies that $\|f_n - f_m\| \le 1$. We have, for $\|x\| = 1$ and $n, m \ge N$,

$$\begin{aligned}
|f(x)| &\le |f(x) - f_N(x)| + |f_N(x)| \\
&= \lim_{n\to+\infty} |f_n(x) - f_N(x)| + |f_N(x)| \\
&\le \limsup_{n\to+\infty} \|f_n - f_N\| + \|f_N\| \le 1 + \|f_N\|,
\end{aligned}$$

so f is bounded, as required.

To finish, we check that f_n converges to f in the norm $\|\cdot\|$ on V^*. Using the fact that, for each $x \in V$, $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence, given $\epsilon > 0$, we may choose $N \ge 1$ such that $n, m \ge N$ implies that

$$|f_n(x) - f_m(x)| \le \epsilon.$$

Letting $m \to +\infty$ gives that, for $n \ge N$,

$$|f_n(x) - f(x)| \le \epsilon,$$

so that

$$\|f_n - f\| = \sup_{\|x\|=1} |f_n(x) - f(x)| \le \epsilon.$$

In other words,

$$\lim_{n\to+\infty} \|f_n - f\| = 0.$$

□