MATH31001/41001/61001: LINEAR ANALYSIS - JANUARY 2010 SOLUTIONS

Examination Feedback

Section A. Apart from **A3**, this section was generally answered quite well. Marks were sometimes lost for mistakes but there were no particular problems. In contrast, **A3** was very badly answered by many students. It was rare for students to recognise they were being asked to sum a geometric series or, if they did, to calculate it correctly. Many students made wildly incorrect assertions: for example, that the series $\sum_{i=1}^{\infty} 2^{-2i}$ converged because its ith terms tended to zero or, even worse, that its sum was equal to zero. This shocked a shocking lack of familiarity and confidence with basic material on infinite series.

Section B. Pleasingly, none of the four questions was particularly avoided by students. Apart from a few things I have indicated below, there were no stand out problems to comment on.

B8 Nearly everyone got the definitions in parts (i) and (iii) correct. It was rare to get a completely correct proof in part (ii): commonly some bits were missing or confused. A lot of people did part (iv) correctly (or nearly correctly with small slips in carrying out the integration).

B9 Part (i) was often answered correctly and a very large number of students answered part (ii) completely correctly. Part (iii) was found harder, with few students correctly obtaining the lower bound for the norm. (Again, I felt students were often not confident in handling infinite series.)

B10 Part (i) was answered correctly by very many students. Part (ii) was also fairly well answered but some students had forgotten how to get the lower bound. A large number of students could do part (iii) but part (iv) caused more problems: many students could not recall exactly how to show that $(\ell^1)^\perp = \{0\}$ or the direct sum relation.

B11 The definition in part (i) was usually given correctly. Part (ii) caused some problems: often students used $\|a\|_2$ (which doesn't in general exist for $a \in \ell^\infty$) rather than $\|a\|_\infty$ in their bound and often there was a lack of detail in explaining why T_a was self-adjoint only if a was a real vector. This question was often attemped last and I think a lot of people were running out of time towards the end. There was often a lack of logical clarity in explaining why the given eigenvalues in part (iii) were the only ones.

Section C. This section obviously contained harder and unseen material. Many students obtained good marks on part (i) and, particularly, part (ii). Pleasingly, quite a few students made serious attempts on the unseen parts, often scoring good marks.

Typeset by $\mathcal{AMS}$-TEX

A1. $\|\cdot\|$ is a norm on a vector space V if it is a function $\|\cdot\| : V \to \mathbb{R}$ such that, for all $x, y \in V$ and all scalars λ,

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda| \, \|x\|$;
(3) $\|x + y\| \leq \|x\| + \|y\|$.

[3 marks]

A2.

$$\ell^2 = \left\{ x = (x_1, x_2, x_3, \dots) : \; x_i \in \mathbb{C}, \; i \geq 1, \; \text{and} \; \sum_{i=1}^{\infty} |x_i|^2 < +\infty \right\}.$$

Let e_n denote the element of ℓ^2 with 1 in the nth place and zero elsewhere. Clearly, for arbitrary $n_1, \dots, n_m$,

$$\sum_{k=1}^{m} \lambda_{n_k} e_{n_k} = 0 \qquad \Longrightarrow \qquad \lambda_{n_1} = \cdots \lambda_{n_m} = 0,$$

so e_n is an infinite linearly independent set. Thus ℓ^2 is infinite dimensional.

[4 marks]

A3. The norm $\|\cdot\|_2$ is defined by

$$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}.$$

We have

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \frac{1/4}{1 - (1/4)} = \frac{1}{3},$$

which is finite, so $x \in \ell^2$.

From above,

$$\|x\|_2 = \frac{1}{\sqrt{3}}.$$

[6 marks]

A4.

$$L^{\perp} = \{x \in H : \langle x, y\rangle = 0, \text{ for all } y \in L\}.$$

[2 marks]

A5. The norm $\|f\|$ of a bounded linear functional $f : V \to \mathbb{C}$ is defined by (either formula gets full marks)

$$\|f\| = \sup_{\substack{x\in V\\ x\neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in V\\ \|x\|=1}} |f(x)|.$$

[2 marks]

A6. The dual space V^* is the space of all bounded linear functionals $f : V \to \mathbb{C}$.
The second dual V^{**} is the dual space of V^*.
Define $i : V \to V^{**}$ by

$$i(x)(f) = f(x) \quad (\text{where } f \in V^*).$$

Then V is reflexive if i is an isometric isomorphism.

[4 marks]

A7.(i) The spectrum of T is the set

$$\text{spec}(T) = \{z \in \mathbb{C} : (zI - T) : V \to V \text{ is not invertible }\}.$$

(ii) The spectral radius of T is the quantity

$$\rho(T) = \sup\{|z| : z \in \text{spec}(T)\}.$$

[4 marks]

B8.(i) A metric space is complete if every Cauchy sequence converges.

[2 marks]

(ii) Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence for $\|\cdot\|_\infty$. Then, given $\epsilon > 0$, there exists $N \geq 1$ such that, for $n, m \geq N$ and for any $x \in [0,1]$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon. \qquad (*)$$

Hence, for eacy $x \in [0,1]$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and so, since $\mathbb{R}$ is complete, it has a limit $f(x)$, say.

Let $m \to +\infty$ in $(*)$ to obtain, for $n \geq N$,

$$|f_n(x) - f(x)| = \lim_{m\to+\infty} |f_n(x) - f_m(x)| \leq \limsup_{m\to+\infty} \|f_n - f_m\|_\infty \leq \epsilon.$$

Hence, $f_n(x)$ converges unifomly to $f(x)$ and so f is continuous.

Taking the supremum over $x \in [0,1]$ in the inequality gives that, for $n \geq N$,

$$\|f_n - f\|_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)| \leq \epsilon,$$

so $\lim_{n\to+\infty} \|f_n - f\|_\infty = 0$, giving convergence in the required norm.

[12 marks]

(iii) Two norms $\|\cdot\|$ and $\|\cdot\|'$ on V are equivalent if there exist $0 < C_1 < C_2$ such that

$$C_1\|x\| \leq \|x\|' \leq C_2\|x\|, \quad \text{for all } x \in V.$$

[3 marks]

(iv) Consider, for example, a sequence $f_n \in C([0,1],\mathbb{R})$, $n \geq 1$, defined by

$$f_n(x) = \begin{cases} (1-nx)^{1/2} & 0 \leq x < 1/n \\ 0 & 1/n \leq x \leq 1 \end{cases}.$$

Then

$$\|f_n\|_\infty = \sup_{x\in[0,1]} |f_n(x)| = 1 \quad \text{for all } n \geq 1$$

but

$$\begin{aligned}\|f_n\|_2 &= \left(\int_0^1 (f(x))^2 dx\right)^{1/2} = \left(\int_0^{1/n} (1-nx)\ dx\right)^{1/2} \\ &= \left(\left[x - \frac{nx^2}{2}\right]_0^{1/n}\right)^{1/2} = \left(\frac{1}{n} - \frac{1}{2n}\right)^{1/2} \\ &= \frac{1}{\sqrt{2n}} \to 0,\end{aligned}$$

as $n \to +\infty$. In particular, there is no $C_2 > 0$ such that

$$\|f_n\|_\infty \le C_2 \|f_n\|_2 \quad \text{for all } n \ge 1,$$

so $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are not equivalent. [Many students used $1 - nx$ instead of $(1-nx)^{1/2}$ – this is completely fine but it makes the calculation of $\|f_n\|_2$ a little more complicated.]

Another possible solution (out of infinitely many!) is to take

$$f_n(x) = \begin{cases} (n - n^2 x)^{1/2} & 0 \le x < 1/n \\ 0 \quad 1/n \le x \le 1 \end{cases}.$$

Then

$$\|f_n\|_2 = \left(\int_0^{1/n} (n - n^2 x)\, dx \right)^{1/2} = \frac{1}{\sqrt{2}}, \quad \text{for all } n \ge 1$$

but

$$\|f_n\|_\infty = \sup_{x \in [0,1]} |f_n(x)| = \sqrt{n} \to +\infty, \text{ as } n \to +\infty.$$

In particular, $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are not equivalent.

[8 marks]

B9.(i) f bounded means there exists $M > 0$ such that

$$|f(x)| \leq M \ \|x\|, \quad \text{for all } x \in V.$$

Suppose $x \in V$ and $\epsilon > 0$. Choose

$$\delta = \frac{\epsilon}{M}.$$

If $y \in V$ with $\|x - y\| < \delta$ then

$$|f(x) - f(y)| = |f(x - y)| \leq M\|x - y\| < M\delta = \epsilon,$$

so f is continuous.

[5 marks]

(ii) Linearity of f follows from standard properties of the integral.

For $\phi \in C([0,1], \mathbb{R})$, we have

$$|f(\phi)| = \left| \int_0^1 \sqrt{x}\phi(x)dx \right| \leq \left| \int_0^1 \sqrt{x}dx \right| \|\phi\|_\infty = \frac{2}{3}\|\phi\|_\infty.$$

This shows that f is bounded and that

$$\|f\| \leq \frac{2}{3}.$$

Now take $\phi = 1$, so $\|\phi\|_\infty = 1$. Then

$$|f(1)| = \left| \int_0^1 \sqrt{x}dx \right| = \frac{2}{3},$$

so

$$\|f\| = \sup_{\|\phi\|_\infty = 1} |f(\phi)| \geq \frac{2}{3}.$$

Thus

$$\|f\| = \frac{2}{3}.$$

[10 marks]

(iii) Since, for $x \in \ell^1$,

$$|g(x)| = \left| \sum_{i=1}^\infty \left(3 - \frac{1}{i}\right) x_i \right| \leq \sum_{i=1}^\infty \left(3 - \frac{1}{i}\right) |x_i| \leq 3 \sum_{i=1}^\infty |x_i| = 3\|x\|_1,$$

we see that g is well defined, bounded and $\|g\| \leq 3$. Linearity is immediate from the definition.

Given $\epsilon > 0$, choose j so that $1/j < \epsilon$ and define $y \in \ell^1$ by

$$y_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Then

$$\|y\|_1 = 1$$

and

$$|g(y)| = \left| \sum_{i=1}^{\infty} \left(3 - \frac{1}{i}\right) y_i \right| = \left(3 - \frac{1}{j}\right) > 3 - \epsilon,$$

so that

$$\|g\| \geq \sup_{\|x\|_1 = 1} |g(x)| > 3 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\|g\| \geq 3$$

and hence

$$\|g\| = 3.$$

[10 marks]

B10.(i) If $x = 0$ or $y = 0$, the result is immediate.
Suppose $x \neq 0$ and $y \neq 0$ and put

$$\lambda = -\frac{\overline{\langle x, y\rangle}}{\langle x, x\rangle}.$$

Then

$$\begin{aligned} 0 &\leq \langle \lambda x + y, \lambda x + y\rangle \\ &= \left|\frac{\overline{\langle x, y\rangle}}{\langle x, x\rangle}\right|^2 \langle x, x\rangle - \frac{\overline{\langle x, y\rangle}\langle x, y\rangle}{\langle x, x\rangle} - \frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle} + \langle y, y\rangle \\ &= \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} - \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} - \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} + \langle y, y\rangle \\ &= -\frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} + \langle y, y\rangle \end{aligned}$$

(using $\langle y, x\rangle = \overline{\langle x, y\rangle}$). Hence

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle,$$

as required.

[8 marks]

(ii) Linearity of f_y follows immediately from properties of the inner product.
For $x \in H$,

$$|f_y(x)| = |\langle x, y\rangle| \leq \|x\| \ \|y\|,$$

using the Cauchy-Schwarz inequality. This shows that f_y is bounded and that $\|f_y\| \leq \|y\|$.
Putting $x = y$, gives

$$|f_y(y)| = |\langle y, y\rangle| = \|y\|^2,$$

so

$$\|f_y\| = \sup_{x \neq 0} \frac{|f_y(x)|}{\|x\|} \geq \frac{|f_y(y)|}{\|y\|} = \|y\|.$$

Hence

$$\|f_y\| = \|y\|.$$

[6 marks]

(iii) The functional f is of the form f_y, where

$$y = (2, -3, 0, 0, \dots).$$

Thus

$$\|f\| = \|y\|_2 = (|2|^2 + |-3|^2)^{1/2} = \sqrt{13}.$$

[4 marks]

(iv) We have

$$(\ell^1)^\perp = \left\{ y \in \ell^2 \,:\, \langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} = 0 \ \forall x \in \ell^1 \right\}.$$

Let $e_j \in \ell^1$ denote the vector with 1 in the jth place and zero elsewhere. Then, for each $j \geq 1$,

$$y \in (\ell^1)^\perp \quad \Longrightarrow \quad \langle e_j, y \rangle = y_j = 0,$$

so

$$(\ell^1)^\perp = \{0\}.$$

Since we always have

$$H = \overline{L} \oplus L^\perp,$$

we conclude that

$$\ell^2 = \overline{\ell^1},$$

i.e., ℓ^1 is dense in ℓ^2 wrt $\| \cdot \|_2$.
[7 marks]

B11.(i) $T : H \to H$ is self-adjoint if $T^* = T$, where the adjoint T^* is defined by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall x, y \in H.$$

[3 marks]

(ii) The linearity of T_a is clear.

For $x = (x_1, x_2, x_3, \dots) \in \ell^2$, we have

$$\|T_a(x)\|_2^2 = \sum_{i=1}^{\infty} |a_i x_i|^2 \le \|a\|_\infty^2 \sum_{i=1}^{\infty} |x_i|^2 = \|a\|_\infty^2 \|x\|_2^2,$$

so that

$$\|T_a(x)\|_2 \le \|a\|_\infty \|x\|_2.$$

This shows that T_a is bounded.

If a is a real vector, we have

$$\langle x, Ty \rangle = \sum_{i=1}^{\infty} x_i \overline{(a_i y_i)} = \sum_{i=1}^{\infty} (a_i x_i) \overline{y_i} = \langle Tx, y \rangle,$$

for all $x, y \in \ell^2$, so that T is self-adjoint.

If a is not real and the entry $a_j \in \mathbb{C} \backslash \mathbb{R}$, say, then choose

$$x = y = e_j,$$

the vector with entry 1 in the jth place and 0 elsewhere. We have

$$\langle x, Ty \rangle = \overline{a_j} \ne a_j \langle Tx, y \rangle,$$

so that T_a is not self-adjoint.

[10 marks]

(iii)For $x = (x_1, x_2, x_3, \dots) \in \ell^2$, we have

$$\|S_n(x)\|_2 = |x_n| \le \|x\|_2.$$

This show that $\|S_n\| \le 1$.

Now for each n, we have

$$\|S_n(e_n)\| = 1 = \|e_n\|_2,$$

so that $\|S_n\| \ge 1$. This shows that $\|S_n\| = 1$.

[5 marks]

(a) Suppose that $(S_1 - \lambda I)(x) = 0$. Then

$$(x_1 - \lambda x_1, -\lambda x_2, -\lambda x_3, \dots) = 0.$$

For $x_1 \neq 0$, $(x_1, 0, 0, \dots)$ gives a solution $\lambda = 1$.
For $x_2 \neq 0$, $(0, x_2, 0, \dots)$ gives a solution $\lambda = 0$.
For $\lambda \neq 0, 1$, $x_1 = \lambda x_1$ forces $x_1 = 0$ and, for $j \geq 2$, $-\lambda x_j = 0$ forces $x_j = 0$. Therefore there are no other eigenvalues.

(b) For $n \geq 2$, suppose that $(S_n - \lambda I)(x) = 0$. Then

$$(x_n - \lambda x_1, -\lambda x_2, -\lambda x_3, \dots) = 0.$$

For $x_1 \neq 0$, $(x_1, 0, 0, \dots)$ gives a solution $\lambda = 0$.
For $\lambda \neq 0$, $-\lambda x_j = 0$ forces $x_j = 0$, for $j \geq 2$. In particular, $x_n = 0$. Then $x_n = \lambda x_1$ forces $x_1 = 0$. Therefore there are no other eigenvalues.

[7 marks]

C12.(i) The set

$$C_N = \{(x_i)_{i=1}^{\infty} \mid x_i \in \mathbb{Q},\ x_i = 0 \text{ for } i \geq N\}$$

is countable and therefore so is

$$C = \bigcup_{N=0}^{\infty} C_N.$$

Suppose that $(y_i)_{i=1}^{\infty} \in \ell^1(\mathbb{R})$ and choose $\epsilon > 0$. By definition, there exists $N \geq 0$ such that

$$\sum_{i=N}^{\infty} |y_i| < \epsilon.$$

Also, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $(x_i)_{i=1}^{\infty} \in C_N$ such that

$$|y_i - x_i| < \frac{\epsilon}{2^i},\ i = 1, \ldots, N-1.$$

Then

$$\|(y_i)_{i=1}^{\infty} - (x_i)_{i=1}^{\infty}\|_1 = \sum_{i=1}^{N-1} |y_i - x_i| + \sum_{i=N}^{\infty} |y_i| < \epsilon + \sum_{i=1}^{N-1} \frac{\epsilon}{2^i} < 2\epsilon.$$

Therefore C is dense in $\ell^1(\mathbb{R})$.
[10 marks]

(ii) If $f = 0$ they we may take $\widetilde{f} = 0$ to be the extension, so suppose that $f \neq 0$ and hence that $\|f\| > 0$. Without loss of generality, assume $\|f\| = 1$.

Any vector in $\operatorname{span}(\{W, x_0\})$ is of the form

$$w + \lambda x_0, \quad \text{with } w \in W \text{ and } \lambda \in \mathbb{R}.$$

Since $\widetilde{f}$ has to be linear, we must define it to be

$$\widetilde{f}(w + \lambda x_0) = \widetilde{f}(w) + \lambda r_0 = f(w) + \lambda r_0,$$

for some choice of $r_0 \in \mathbb{R}$.

So that we do not increase the norm $\|f\| = 1$, choose r_0 so that

$$|\widetilde{f}(w + \lambda x_0)| \leq \|w + \lambda x_0\|, \quad \text{for all } \lambda \in \mathbb{R}, w \in W,$$

i.e.,

$$-\|w + \lambda x_0\| \leq \widetilde{f}(w + \lambda x_0) \leq \|w + \lambda x_0\|,$$

and hence as

$$-f(w) - \|w + \lambda x_0\| \leq \lambda r_0 \leq -f(w) + \|w + \lambda x_0\|.$$

For $\lambda \neq 0$, this becomes

$$-f\left(\frac{w}{\lambda}\right) - \left\|x_0 + \frac{w}{\lambda}\right\| \leq r_0 \leq -f\left(\frac{w}{\lambda}\right) + \left\|x_0 + \frac{w}{\lambda}\right\|.$$

However w/λ is an arbitrary element of W, so we may rewrite as

$$-f(w) - \|x_0 + w\| \leq r_0 \leq -f(w) + \|x_0 + w\|, \qquad \text{for all } w \in W.$$

To show that such an r_0 can be chosen, we need to show that

$$\sup_{w_1 \in W} (-f(w_1) - \|w_1 + x_0\|) \leq \inf_{w_2 \in W} (-f(w_2) + \|w_2 + x_0\|).$$

However, if $w_1, w_2 \in W$ then (using $\|f\| = 1$)

$$\begin{aligned} f(w_2) - f(w_1) &\leq |f(w_2) - f(w_1)| \\ &\leq \|w_2 - w_1\| = \|(w_2 + x_0) - (w_1 + x_0)\| \\ &\leq \|w_2 + x_0\| + \|w_1 + x_0\|, \end{aligned}$$

so that

$$-f(w_1) - \|w_1 + x_0\| \leq -f(w_2) + \|w_2 + x_0\|.$$

This gives the required inequality. So we only need to choose $r_0 \in [A, B]$ to ensure that $\|\widetilde{f}\| = \|f\|$.
[20 marks]

(iii) By part (ii), f may be extended to a bounded linear functional

$$f_1 : W_1 \to \mathbb{R}$$

such that $\|f_1\| = \|f\|$.

Proceed by induction. Suppose that $f_n : W_n \to \mathbb{R}$ is defined with the desired properties.

If $e_n \in W$ then $W_{n+1} = W_n$ and we define $f_{n+1} = f_n$. Then

$$\|f_{n+1}\| = \|f_n\| = \|f\|.$$

Also, if $x \in W$ then $x \in W_n$, so

$$f_{n+1}(x) = f_n(x) = f(x),$$

so f_{n+1} is an extension of f.

If $e_n \notin W$ then use part (iii) to obtain a linear functional $f_{n+1} : W_{n+1} \to \mathbb{R}$ again with

$$\|f_{n+1}\| = \|f_n\| = \|f\|.$$

Also, if $x \in W$ then $x \in W_n$. Since f_{n+1} is an extension of f_n, we have

$$f_{n+1}(x) = f_n(x) = f(x),$$

so f_{n+1} is an extension of f.

[Full marks awarded even if the trivial case $e_n \in W$ is omitted in the induction.]

[10 marks]

(iv) For $x, y \in \ell^1(\mathbb{R})$ and $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
F(ax+by) &= \lim_{n\to+\infty} f_n(ax_1+by_1, ax_2+by_2, \dots, ax_n+by_n, 0, 0, \dots) \\
&= \lim_{n\to+\infty} (af_n(x_1, x_2, \dots, x_n, 0, 0, \dots) + bf_n(y_1, y_2, \dots, y_n, 0, 0, \dots)) \\
&= a \lim_{n\to+\infty} f_n(x_1, x_2, \dots, x_n, 0, 0, \dots) + b \lim_{n\to+\infty} f_n(y_1, y_2, \dots, y_n, 0, 0, \dots) \\
&= aF(x) + bF(y),
\end{aligned}$$

which shows that F is linear.

For $x \in \ell^1(\mathbb{R})$, we have

$$|F(x)| = \lim_{n\to+\infty} |f_n(x_1, x_2, \dots, x_n, 0, 0, \dots)| \leq \lim_{n\to+\infty} \|f_n\|\|x\|_1 = \|f\|\|x\|_1.$$

This shows that F is bounded and that $\|F\| = \|f\|$. Taking $x \in W$ shows that $\|F\| = \|f\|$.
[10 marks]