MATH31001/MATH41001/MATH61001: LINEAR ANALYSIS

Revision Checklist

The purpose of the document is to help you focus your revision for the MATH31001 and MATH41001/MATH61001 exams by highlighting key topics. Topics only relevant for MATH41001/MATH61001 are in square brackets [$\cdots$].

Examination Format. The examination with have the same format as the mock exam. *MATH31001:* Answer all seven questions in Section A (25 marks in total) and three of the four questions in Section B (25 marks each). The questions will be a mixture of bookwork and examples similar to those you have seen in lectures/examples sheets.
MATH41001/MATH61001: Answer all seven questions in Section A (25 marks in total) and three of the four questions in Section B (25 marks each) and all of Section C (50 marks). The mark out of 150 will be scaled to give a mark out of 100. Section A and B questions identical to those for MATH31001. Section C will consist of bookwork based on the extra reading, previously seen examples (which are not necessarily restricted to the extra reading) and unseen examples.

Health Warning. A particular topic being mentioned or not mentioned below does not guarantee that it will or will not appear on the examination.

Infinite dimensional vector spaces. Definition of vector space, linear independence, span, dimension. Examples: $\mathbb{R}^n$, $\mathbb{C}^n$, $C([0,1],\mathbb{R})$, $R([0,1],\mathbb{R})$ (= Riemann integrable functions on $[0,1]$), $\ell^1, \ell^1(\mathbb{R})$, $\ell^2, \ell^2(\mathbb{R})$, more generally $\ell^p, \ell^p(\mathbb{R})$, $1 \leq p < +\infty$, $\ell^\infty, \ell^\infty(\mathbb{R})$. The vectors e_n (with 1 in the nth place, 0 elsewhere), $n \geq 1$. Proof that (apart from $\mathbb{R}^n, \mathbb{C}^n$) these examples are infinite dimensional. Proof that $\ell^p \subset \ell^q$ but $\ell^p \neq \ell^q$, for $1 \leq p < q \leq +\infty$.

Norms on vector spaces. Definition of a norm. Definition of $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p, \|\cdot\|_\infty$ on $\mathbb{R}^n, \mathbb{C}^n$. Cauchy-Schwarz Inequality to prove $\|\cdot\|_2$ is a norm. [Proof that $\|\cdot\|_p$ is a norm using Hölder's Inequality and Minkowski's Inequality] The norms $\|\cdot\|_p$ on $\ell^p, \ell^p(\mathbb{R})$; calculate norms for simple examples by finding the sum of an infinite series.

Definition of equivalent norms. The norms $\|\cdot\|_\infty, \|\cdot\|_1, \|\cdot\|_2$ on $C([0,1],\mathbb{R})$; proof that that these are *not* equivalent. Proof that polynomials are dense in $C([0,1],\mathbb{R})$ wrt the norm $\|\cdot\|_\infty$ (Weierstrass Approximation Theorem). [Definition of an algebra of functions. Stone-Weierstrass Theorem.]

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Banach spaces. Definitions of Cauchy sequence, completeness, Banach space. Proof that $(C([0,1],\mathbb{R}),\|\cdot\|_\infty)$, $(\ell^1,\|\cdot\|_1)$ and $(\ell^1(\mathbb{R}),\|\cdot\|_1)$ are Banach spaces. [Proof that $(\ell^p,\|\cdot\|_p)$, $1<p<+\infty$ and $(\ell^\infty,\|\cdot\|_\infty)$ are Banach spaces.] Proof that $(C([0,1],\mathbb{R}),\|\cdot\|_1)$, $(C([0,1],\mathbb{R}),\|\cdot\|_2)$ are *not* Banach spaces. Definition of a separable metric space. Pr oof that $C([0,1],\mathbb{R})$ is separable (wrt $\|\cdot\|_\infty$). Proof that $\ell^p,\ell^p(\mathbb{R})$, $1\le p<+\infty$, are separable. Proof that $\ell^\infty,\ell^\infty(\mathbb{R})$ are *not* separable.

Hilbert spaces. Definition of inner product. Proof of Cauchy-Schwarz Inequality for inner products and proof that the inner product defines a norm. Examples, particularly $\ell^2,\ell^2(\mathbb{R})$. Orthogonal complements and orthogonal decomposition (see Lemma 2.12). Calculation of orthogonal complements for simple examples.

Linear functionals. Definition of linear functional and definition of bounded linear functional. Proof that bounded is equivalent to continuous in this context. Definition of the norm of a bounded linear functional. Definition of the dual space V^*. You should be able to show that given examples of linear functionals on $C([0,1],\mathbb{R})$ and ℓ^p, $1\le p<+\infty$, are bounded and calculate their norms. Definition of isometric isomorphism. Riesz Representation Theorem. [Dual spaces for l^p, $1<p<+\infty$, and ℓ^1.] Statement of the Hahn-Banach Theorem. [Proof of the Hahn-Banach Theorem. All of it.] Definition of the second dual V^{**} and the isometric embedding of V in V^{**} (Lemma 3.9).

Linear operators. Definition of linear operator and definition of bounded linear operator. Definition of the norm of a bounded linear operator: be able to show that given examples are bounded and calculate the norm. The shift operator on ℓ^p. The adjoint operator on Hilbert spaces. Calculation of the adjoint for given examples. Definition of self-adjoint: showing that examples are or are not self-adjoint.

Definition of invertibility of an operator in $B(V)$. Definition of eigenvalue. Definition of spectrum. Proof that the spectrum is closed and contained in $\{z\in\mathbb{C}:|z|\le\|T\|\}$. Definition of spectral radius and spectral radius formula. Calculations of eigenvalues, spectral radius, spectrum in given examples.