

Three hours

UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

Mock Exam

Answer **ALL** of the seven questions in Section A and **THREE** of the four questions in Section B and **ALL** of Section C.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. What is meant by saying that $d : X \times X \rightarrow \mathbb{R}$ is a *metric* on a set X ?

[3 marks]

A2. State the definition of the space ℓ^1 and show that it is infinite dimensional.

[4 marks]

A3. Define $x = (x_i)_{i=1}^{\infty}$ by

$$x = \left(1, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{4}}, \dots, \frac{(-1)^{i+1}}{\sqrt{i}}, \dots \right).$$

For which values of $p \in [1, \infty]$ do we have $x \in \ell^p$? Justify your answer.

[4 marks]

A4. Let H be a vector space over \mathbb{C} . What is meant by saying that a function $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$ is an *inner product*?

[3 marks]

A5. Let V be a Banach space over \mathbb{C} . State the definition the *dual space* V^* .

For $f \in V^*$, what is meant by its *norm* $\|f\|$?

[4 marks]

A6. What is meant by the term *isometric isomorphism*?

Let H be a Hilbert space over \mathbb{C} . Write down an isometric isomorphism between H and H^* .

[3 marks]

A7. Let V be a Banach space over \mathbb{C} and let $T : V \rightarrow V$ be a bounded linear operator. State what is meant by

(i) an *eigenvalue* of T ;

(ii) the *spectrum* of T ?

[4 marks]

SECTION BAnswer **THREE** of the four questions**B8.**

- (i) State what is meant when we say that a normed vector space is a *Banach space*.
(ii) Show that the vector space ℓ^1 with the norm

$$\|(x_i)_{i=1}^{\infty}\|_1 = \sum_{i=1}^{\infty} |x_i|$$

is a *Banach space*.(You may assume that \mathbb{C} is complete.)

- (iii) Consider the norm $\|\cdot\|_2$ on ℓ^1 defined by

$$\|(x_i)_{i=1}^{\infty}\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}.$$

Show that ℓ^1 with the norm $\|\cdot\|_2$ is *not* a Banach space.(You do not need to prove that $\|\cdot\|_2$ is a norm on ℓ^1 .)

[25 marks]

B9.

- (i) Let V be a normed vector space over \mathbb{R} and let $f : V \rightarrow \mathbb{R}$ be a linear functional. Show that $f : V \rightarrow \mathbb{R}$ is continuous at 0 then it is continuous at every $x \in V$.
- (ii) Let $V = C([0, 1], \mathbb{R})$ with the norm

$$\|\phi\|_{\infty} = \sup_{x \in [0, 1]} |\phi(x)|.$$

Define $f : V \rightarrow \mathbb{R}$ by

$$f(\phi) = \int_0^1 \sin(\pi x) \phi(x) dx.$$

Show that f is a bounded linear functional and calculate its norm $\|f\|$.

- (iii) Let $V = \ell^1$ with the norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|,$$

where $x = (x_1, x_2, x_3, \dots)$. Define $g : V \rightarrow \mathbb{R}$ by

$$g(x_1, x_2, x_3, \dots) = \sum_{i=1}^{\infty} \frac{i}{i+1} x_i.$$

Show, directly from its definition, that g is a bounded linear functional and show that $\|g\| = 1$.

[25 marks]

B10.

(i) Let H be a Hilbert space over \mathbb{C} with inner product $\langle \cdot, \cdot \rangle$. State *Cauchy-Schwarz inequality*.

(ii) Show that the formula

$$\|x\| = \langle x, x \rangle^{1/2}$$

defines a norm on H .

(iii) Let H be a Hilbert space over \mathbb{C} and let L be a linear subspace of H . State what is meant by the *orthogonal complement* L^\perp of L ?

(iv) Let $H = \ell^2$ and let

$$L = \{(x_1, x_2, x_3, \dots) \in \ell^2 : x_1 = 0\}.$$

(a) Show that L is a closed subspace of ℓ^2 .

(b) Calculate the orthogonal complement L^\perp .

(c) Show directly that $\ell^2 = L \oplus L^\perp$.

[25 marks]

B11.

(i) Let H be a Hilbert space over \mathbb{C} . State what is meant by a the *adjoint* of a linear operator $T : H \rightarrow H$.

(ii) Let $H = \ell^2$ and define $T : \ell^2 \rightarrow \ell^2$ to be the shift operator

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

(a) Show that T is a bounded linear operator and calculate its norm $\|T\|$.

(b) Calculate an explicit expression for the adjoint operator T^* .

(iii) Let $H = \ell^2$ and define $S : \ell^2 \rightarrow \ell^2$ by

$$S(x_1, x_2, x_3, \dots, x_i, \dots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_i}{i}, \dots \right).$$

(a) Calculate an explicit expression for the adjoint operator S^* .

(b) Calculate $\|S^n\|$, for all $n \geq 1$, and hence calculate the spectral radius of S .

You may assume Stirling's formula:

$$\lim_{n \rightarrow +\infty} \frac{n!}{n^{n+1/2} e^{-n}} = \sqrt{2\pi}.$$

[25 marks]

SECTION CAnswer **ALL** of this section**C12.** Recall that a metric space is *separable* if it contains a countable dense subset.

- (i) Show that $\ell^\infty(\mathbb{R})$ is *not* separable.
- (ii) Determine whether or not the space

$$c_0(\mathbb{R}) = \{(x_i)_{i=1}^\infty : x_i \in \mathbb{R}, \lim_{i \rightarrow +\infty} x_i = 0\},$$

with the norm $\|\cdot\|_\infty$, is separable, justifying your answer.Let V be a vector space over \mathbb{R} and let $A \subset V$ be an arbitrary subset. Recall that the span of A , $\text{span}(A)$, is the set of all *finite* linear combinations

$$\lambda_1 x_1 + \cdots + \lambda_m x_m,$$

where $x_1, \dots, x_m \in A$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.

- (iii) For $n \geq 1$, let $e_n \in \ell^1(\mathbb{R})$ denote the vector whose entries are 1 in the n th place and 0 elsewhere. Show that $\text{span}(\{e_n\}_{n=1}^\infty)$ is dense in $\ell^1(\mathbb{R})$.
- (iv) Let V be a vector space over \mathbb{R} with norm $\|\cdot\|$ and let W be a linear subspace of V . Prove the following result:
Suppose that $f : W \rightarrow \mathbb{R}$ is a bounded linear functional and that $x_0 \in V \setminus W$. Show that f extends to a bounded linear functional $\tilde{f} : \text{span}(\{W, x_0\}) \rightarrow \mathbb{R}$ with $\|\tilde{f}\| = \|f\|$.
- (v) State (but do not prove) the Hahn-Banach Theorem.

[50 marks]

END OF EXAMINATION PAPER