

Three hours

UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

Answer **ALL** of the seven questions in Section A and **THREE** of the four questions in Section B and **ALL** of Section C.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. What is meant by saying that $\|\cdot\|$ is a *norm* on a vector space V ?

[3 marks]

A2. State the definition of the space ℓ^2 and show that it is infinite dimensional.

[4 marks]

A3. Define the norm $\|\cdot\|_2$ on the space ℓ^2 .

Define $x = (x_i)_{i=1}^{\infty}$ by

$$x = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^i}, \dots \right).$$

Show that $x \in \ell^2$ and calculate $\|x\|_2$.

[6 marks]

A4. Let H be a Hilbert space over \mathbb{C} and let L be a linear subspace of H . State what is meant by the *orthogonal complement* L^\perp of L .

[2 marks]

A5. Let V be a Banach space over \mathbb{C} . State what is meant by the *norm* of a bounded linear functional $f : V \rightarrow \mathbb{C}$.

[2 marks]

A6. Let V be a Banach space over \mathbb{C} . State the definitions of the *dual space* V^* and the *second dual* V^{**} . What is meant by saying that V is *reflexive*?

[4 marks]

A7. Let V be a Banach space and let $T : V \rightarrow V$ be a bounded linear operator. State what is meant by

(i) the *spectrum* of T ;

(ii) the *spectral radius* of T .

[4 marks]

SECTION BAnswer **THREE** of the four questions**B8.**

- (i) State what is meant when we say that a metric space is *complete*.
- (ii) Let $C([0, 1], \mathbb{R})$ denote the vector space of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. Show that $C([0, 1], \mathbb{R})$ with the norm

$$\|f\|_{\infty} = \sup_{x \in [0, 1]} |f(x)|$$

is a *Banach space*.

(You may assume that \mathbb{R} is complete and that the uniform limit of a sequence of continuous functions is continuous.)

- (iii) State what is meant when we say that two norms on a vector space are *equivalent*.
- (iv) Define a norm $\|\cdot\|_2$ on $C([0, 1], \mathbb{R})$ by

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}.$$

Show that $\|\cdot\|_2$ is *not* equivalent to $\|\cdot\|_{\infty}$.

(You do not need to prove that $\|\cdot\|_2$ is a norm.)

[25 marks]

B9.

- (i) Let V be a normed vector space over \mathbb{R} and let $f : V \rightarrow \mathbb{R}$ be a bounded linear functional. Show that $f : V \rightarrow \mathbb{R}$ is *continuous*.
- (ii) Let $V = C([0, 1], \mathbb{R})$ with the norm

$$\|\phi\|_{\infty} = \sup_{x \in [0, 1]} |\phi(x)|.$$

Define $f : V \rightarrow \mathbb{R}$ by

$$f(\phi) = \int_0^1 \sqrt{x} \phi(x) dx.$$

Show that f is a bounded linear functional and calculate its norm $\|f\|$.

- (iii) Let $V = \ell^1$ with the norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|,$$

where $x = (x_1, x_2, x_3, \dots)$. Define $g : V \rightarrow \mathbb{R}$ by

$$g(x_1, x_2, x_3, \dots) = \sum_{i=1}^{\infty} \left(3 - \frac{1}{i}\right) x_i.$$

Show, directly from its definition, that g is a bounded linear functional and show that $\|g\| = 3$.

[25 marks]

B10.

- (i) Let H be a Hilbert space over \mathbb{C} . Prove the *Cauchy-Schwarz inequality*: for all $x, y \in H$,

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}.$$

- (ii) Let H be a Hilbert space over \mathbb{C} . For $y \in H$, define

$$f_y(x) = \langle x, y \rangle.$$

Show that $f_y : H \rightarrow \mathbb{C}$ is a bounded linear functional and calculate its norm.

- (iii) Let $H = \ell^2$ and, for $x = (x_1, x_2, x_3, \dots) \in \ell^2$, define a linear functional $f : H \rightarrow \mathbb{C}$ by

$$f(x) = 2x_1 - 3x_2.$$

Using part (ii), or otherwise, compute $\|f\|$.

- (iv) Let $H = \ell^2$ and let $L = \ell^1$, regarded as a linear subspace of H . Calculate the orthogonal complement L^\perp in this case.

Deduce that ℓ^1 is dense in ℓ^2 , with respect to the norm $\|\cdot\|_2$.

[25 marks]

B11.

(i) Let H be a Hilbert space over \mathbb{C} . State what is meant by a linear operator $T : H \rightarrow H$ being *self-adjoint*.

(ii) Let $H = \ell^2$ and let $a = (a_1, a_2, a_3, \dots) \in \ell^\infty$. Define

$$T_a(x_1, x_2, x_3, \dots) = (a_1x_1, a_2x_2, a_3x_3, \dots).$$

Show that $T_a : \ell^2 \rightarrow \ell^2$ is a bounded linear operator.

Show that T_a is self-adjoint if and only if a is a real vector (i.e. $a_i \in \mathbb{R}$, for all $i \geq 1$).

(iii) Let $H = \ell^2$ and, for $n \geq 1$, define a sequence of linear operators $S_n : \ell^2 \rightarrow \ell^2$ by

$$S_n(x_1, x_2, x_3, \dots) = (x_n, 0, 0, \dots).$$

Show that $\|S_n\| = 1$, for all $n \geq 1$.

Show that

(a) S_1 has 0 and 1 as eigenvalues and no other eigenvalues;

(b) for $n \geq 2$, S_n has 0 as an eigenvalue and no other eigenvalues.

[25 marks]

SECTION CAnswer **ALL** of this section**C12.** Recall that a metric space is *separable* if it contains a countable dense subset.(i) Show that $\ell^1(\mathbb{R})$ is separable.(ii) Let V be a vector space over \mathbb{R} with norm $\|\cdot\|$ and let W be a linear subspace of V . Prove the following result:Suppose that $f : W \rightarrow \mathbb{R}$ is a bounded linear functional and that $x_0 \in V \setminus W$. Show that f extends to a bounded linear functional $\tilde{f} : \text{span}(\{W, x_0\}) \rightarrow \mathbb{R}$ with $\|\tilde{f}\| = \|f\|$.Now suppose that $V = \ell^1(\mathbb{R})$ and that W is a linear subspace. For $n \geq 1$, let $e_n \in \ell^1(\mathbb{R})$ denote the vector whose entries are 1 in the n th place and 0 elsewhere.Let $W_0 = W$ and, for $n \geq 1$, let $W_n = \text{span}(\{W_{n-1}, e_n\})$.(iii) Assuming the result stated in part (ii), show that, for each $n \geq 1$, $f : W \rightarrow \mathbb{R}$ extends to a bounded linear functional

$$f_n : W_n \rightarrow \mathbb{R},$$

such that $\|f_n\| = \|f\|$.Define $F : \ell^1(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$F((x_i)_{i=1}^{\infty}) = \lim_{n \rightarrow +\infty} f_n(x_1, x_2, \dots, x_n, 0, 0, \dots).$$

(You may assume that the limit exists.)

(iv) Show that F is a bounded linear functional and that $\|F\| = \|f\|$.

[50 marks]

END OF EXAMINATION PAPER