Three hours

UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

Mock Exam

Answer **ALL** of the seven questions in Section A and **THREE** of the four questions in Section B and **ALL** of Section C.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. What is meant by saying that $d : X \times X \to \mathbb{R}$ is a *metric* on a set X?

[3 marks]

A2. State the definition of the space ℓ^1 and show that it is infinite dimensional.

[4 marks]

A3. Define $x = (x_i)_{i=1}^{\infty}$ by

$$x = \left(1, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{-1}{\sqrt{4}}, \ldots, \frac{(-1)^{i+1}}{\sqrt{i}}, \ldots\right).$$

For which values of $p \in [1, \infty]$ do we have $x \in \ell^p$? Justify your answer.

[4 marks]

A4. Let H be a vector space over $\mathbb{C}$. What is meant by saying that a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ is an *inner product*?

[3 marks]

A5. Let V be a Banach space over $\mathbb{C}$. State the definition the *dual space* V^*.

For $f \in V^*$, what is meant by its *norm* $\|f\|$?

[4 marks]

A6. What is meant by the term *isometric isomorphism*?

Let H be a Hilbert space over $\mathbb{C}$. Write down an isometric isomorphism between H and H^*.

[3 marks]

A7. Let V be a Banach space over $\mathbb{C}$ and let $T : V \to V$ be a bounded linear operator. State what is meant by

(i) an *eigenvalue* of T;

(ii) the *spectrum* of T?

[4 marks]

SECTION B

Answer **THREE** of the four questions

B8.

(i) State what is meant when we say that a normed vector space is a *Banach space*.

(ii) Show that the vector space ℓ^1 with the norm

$$\|(x_i)_{i=1}^{\infty}\|_1 = \sum_{i=1}^{\infty} |x_i|$$

is a *Banach space*.

(You may assume that $\mathbb{C}$ is complete.)

(iii) Consider the norm $\|\cdot\|_2$ on ℓ^1 defined by

$$\|(x_i)_{i=1}^{\infty}\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}.$$

Show that ℓ^1 with the norm $\|\cdot\|_2$ is *not* a Banach space.

(You do not need to prove that $\|\cdot\|_2$ is a norm on ℓ^1.)

[25 marks]

B9.

(i) Let V be a normed vector space over $\mathbb{R}$ and let $f : V \to \mathbb{R}$ be a linear functional. Show that $f : V \to \mathbb{R}$ is continuous at 0 then it is continuous at every $x \in V$.

(ii) Let $V = C([0,1], \mathbb{R})$ with the norm

$$\|\phi\|_\infty = \sup_{x\in[0,1]} |\phi(x)|.$$

Define $f : V \to \mathbb{R}$ by

$$f(\phi) = \int_0^1 \sin(\pi x)\phi(x)\ dx.$$

Show that f is a bounded linear functional and calculate its norm $\|f\|$.

(iii) Let $V = \ell^1$ with the norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|,$$

where $x = (x_1, x_2, x_3, \ldots)$. Define $g : V \to \mathbb{R}$ by

$$g(x_1, x_2, x_3, \ldots) = \sum_{i=1}^{\infty} \frac{i}{i+1} x_i.$$

Show, directly from its definition, that g is a bounded linear functional and show that $\|g\| = 1$.

[25 marks]

B10.

(i) Let H be a Hilbert space over $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle$. State *Cauchy-Schwarz inequality.*

(ii) Show that the formula

$$\|x\| = \langle x, x\rangle^{1/2}$$

defines a norm on H.

(iii) Let H be a Hilbert space over $\mathbb{C}$ and let L be a linear subspace of H. State what is meant by the *orthogonal complement* $L^\perp$ of L?

(iv) Let $H = \ell^2$ and let

$$L = \{(x_1, x_2, x_3, \dots) \in \ell^2 \,:\, x_1 = 0\}.$$

(a) Show that L is a closed subspace of ℓ^2.

(b) Calculate the orthogonal complement $L^\perp$.

(c) Show directly that $\ell^2 = L \oplus L^\perp$.

[25 marks]

B11.

(i) Let H be a Hilbert space over $\mathbb{C}$. State what is meant by a the *adjoint* of a linear operator $T : H \to H$.

(ii) Let $H = \ell^2$ and define $T : \ell^2 \to \ell^2$ to be the shift operator

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4 \ldots).$$

(a) Show that T is a bounded linear operator and calculate its norm $\|T\|$.

(b) Calculate an explicit expression for the adjoint operator T^*.

(iii) Let $H = \ell^2$ and define $S : \ell^2 \to \ell^2$ by

$$S(x_1, x_2, x_3, \ldots, x_i, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_i}{i}, \ldots\right).$$

(a) Calculate an explicit expression for the adjoint operator S^*.

(b) Calculate $\|S^n\|$, for all $n \geq 1$, and hence calculate the spectral radius of S.

You may assume Stirling's formula:

$$\lim_{n \to +\infty} \frac{n!}{n^{n+1/2} e^{-n}} = \sqrt{2\pi}.$$

[25 marks]

SECTION C

Answer **ALL** of this section

C12. Recall that a metric space is *separable* if it contains a countable dense subset.

(i) Show that $\ell^\infty(\mathbb{R})$ is *not* separable.

(ii) Determine whether or not the space

$$c_0(\mathbb{R}) = \{(x_i)_{i=1}^\infty : x_i \in \mathbb{R}, \lim_{i\to+\infty} x_i = 0\},$$

with the norm $\|\cdot\|_\infty$, is separable, justifying your answer.

Let V be a vector space over $\mathbb{R}$ and let $A \subset V$ be an arbitrary subset. Recall that the span of A, $\mathrm{span}(A)$, is the set of all *finite* linear combinations

$$\lambda_1 x_1 + \cdots + \lambda_m x_m,$$

where $x_1, \ldots, x_m \in A$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

(iii) For $n \geq 1$, let $e_n \in \ell^1(\mathbb{R})$ denote the vector whose entries are 1 in the nth place and 0 elsewhere. Show that $\mathrm{span}(\{e_n\}_{n=1}^\infty)$ is dense in $\ell^1(\mathbb{R})$.

(iv) Let V be a vector space over $\mathbb{R}$ with norm $\|\cdot\|$ and let W be a linear subspace of V. Prove the following result:

Suppose that $f : W \to \mathbb{R}$ is a bounded linear functional and that $x_0 \in V \setminus W$. Show that f extends to a bounded linear functional $\widetilde{f} : \mathrm{span}(\{W, x_0\}) \to \mathbb{R}$ with $\|\widetilde{f}\| = \|f\|$.

(v) State (but do not prove) the Hahn-Banach Theorem.

[50 marks]

END OF EXAMINATION PAPER