Two hours

UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

Answer **ALL** of the seven questions in Section A and **THREE** of the four questions in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. What is meant by saying that $\|\cdot\|$ is a *norm* on a vector space V?

[3 marks]

A2. State the definition of the space ℓ^2 and show that it is infinite dimensional.

[4 marks]

A3. Define the norm $\|\cdot\|_2$ on the space ℓ^2.

Define $x = (x_i)_{i=1}^{\infty}$ by

$$x = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^i}, \ldots\right).$$

Show that $x \in \ell^2$ and calculate $\|x\|_2$.

[6 marks]

A4. Let H be a Hilbert space over $\mathbb{C}$ and let L be a linear subspace of H. State what is meant by the *orthogonal complement* $L^\perp$ of L.

[2 marks]

A5. Let V be a Banach space over $\mathbb{C}$. State what is meant by the *norm* of a bounded linear functional $f : V \to \mathbb{C}$.

[2 marks]

A6. Let V be a Banach space over $\mathbb{C}$. State the definitions of the *dual space* V^* and the *second dual* V^{**}. What is meant by saying that V is *reflexive*?

[4 marks]

A7. Let V be a Banach space and let $T : V \to V$ be a bounded linear operator. State what is meant by

(i) the *spectrum* of T;

(ii) the *spectral radius* of T.

[4 marks]

SECTION B

Answer **THREE** of the four questions

B8.

(i) State what is meant when we say that a metric space is *complete*.

(ii) Let $C([0,1],\mathbb{R})$ denote the vector space of continuous functions $f : [0,1] \to \mathbb{R}$. Show that $C([0,1],\mathbb{R})$ with the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

is a *Banach space*.

(You may assume that $\mathbb{R}$ is complete and that the uniform limit of a sequence of continuous functions is continuous.)

(iii) State what is meant when we say that two norms on a vector space are *equivalent*.

(iv) Define a norm $\|\cdot\|_2$ on $C([0,1],\mathbb{R})$ by

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2\right)^{1/2} dx.$$

Show that $\|\cdot\|_2$ is *not* equivalent to $\|\cdot\|_\infty$.

(You do not need to prove that $\|\cdot\|_2$ is a norm.)

[25 marks]

B9.

(i) Let V be a normed vector space over $\mathbb{R}$ and let $f : V \to \mathbb{R}$ be a bounded linear functional. Show that $f : V \to \mathbb{R}$ is *continuous*.

(ii) Let $V = C([0,1], \mathbb{R})$ with the norm

$$\|\phi\|_\infty = \sup_{x\in[0,1]} |\phi(x)|.$$

Define $f : V \to \mathbb{R}$ by

$$f(\phi) = \int_0^1 \sqrt{x}\phi(x)\ dx.$$

Show that f is a bounded linear functional and calculate its norm $\|f\|$.

(iii) Let $V = \ell^1$ with the norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|,$$

where $x = (x_1, x_2, x_3, \ldots)$. Define $g : V \to \mathbb{R}$ by

$$g(x_1, x_2, x_3, \ldots) = \sum_{i=1}^{\infty} \left(3 - \frac{1}{i}\right) x_i.$$

Show, directly from its definition, that g is a bounded linear functional and show that $\|g\| = 3$.

[25 marks]

B10.

(i) Let H be a Hilbert space over $\mathbb{C}$. Prove the *Cauchy-Schwarz inequality*: for all $x, y \in H$,

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

(ii) Let H be a Hilbert space over $\mathbb{C}$. For $y \in H$, define

$$f_y(x) = \langle x, y\rangle.$$

Show that $f_y : H \to \mathbb{C}$ is a bounded linear functional and calculate its norm.

(iii) Let $H = \ell^2$ and, for $x = (x_1, x_2, x_3, \ldots) \in \ell^2$, define a linear functional $f : H \to \mathbb{C}$ by

$$f(x) = 2x_1 - 3x_2.$$

Using part (ii), or otherwise, compute $\|f\|$.

(iv) Let $H = \ell^2$ and let $L = \ell^1$, regarded as a linear subspace of H. Calculate the orthogonal complement $L^\perp$ in this case.

Deduce that ℓ^1 is dense in ℓ^2, with respect to the norm $\|\cdot\|_2$.

[25 marks]

B11.

(i) Let H be a Hilbert space over $\mathbb{C}$. State what is meant by a linear operator $T : H \to H$ being *self-adjoint*.

(ii) Let $H = \ell^2$ and let $a = (a_1, a_2, a_3, \ldots) \in \ell^\infty$. Define

$$T_a(x_1, x_2, x_3, \ldots) = (a_1x_1, a_2x_2, a_3x_3, \ldots).$$

Show that $T_a : \ell^2 \to \ell^2$ is a bounded linear operator.

Show that T_a is self-adjoint if and only if a is a real vector (i.e. $a_i \in \mathbb{R}$, for all $i \geq 1$).

(iii) Let $H = \ell^2$ and, for $n \geq 1$, define a sequence of linear operators $S_n : \ell^2 \to \ell^2$ by

$$S_n(x_1, x_2, x_3, \ldots) = (x_n, 0, 0, \ldots).$$

Show that $\|S_n\| = 1$, for all $n \geq 1$.

Show that

(a) S_1 has 0 and 1 as eigenvalues and no other eigenvalues;

(b) for $n \geq 2$, S_n has 0 as an eigenvalue and no other eigenvalues.

[25 marks]

END OF EXAMINATION PAPER