MATH31001/41001/61001: LINEAR ANALYSIS - MOCK EXAM SOLUTIONS

A1. $d : X \times X \to \mathbb{R}$ is a metric on X if, for all $x, y, z \in X$,

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

[3 marks]

A2.

$$\ell^1 = \left\{ x = (x_1, x_2, x_3, \dots) : x_i \in \mathbb{C},\ i \geq 1, \text{ and } \sum_{i=1}^{\infty} |x_i| < +\infty \right\}.$$

Let e_n denote the element of ℓ^1 with 1 in the nth place and zero elsewhere. Clearly, for arbitrary $n_1, \dots, n_m$,

$$\sum_{k=1}^{m} \lambda_{n_k} e_{n_k} = 0 \qquad \Longrightarrow \qquad \lambda_{n_1} = \cdots \lambda_{n_m} = 0,$$

so e_n is an infinite linearly independent set. Thus ℓ^1 is infinite dimensional.

[4 marks]

A3. First consider finite $p \geq 1$. We have

$$\sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{\infty} \frac{1}{i^{p/2}}.$$

This converges for $p/2 > 1$, i.e., $p > 2$, and diverges for $p/2 \leq 1$, i.e., $p \leq 2$. So $x \in \ell^p$ if and only if $p > 2$.

Now

$$\sup_{i \geq 1} |x_i| = \sup_{i \geq 1} \frac{1}{\sqrt{i}} = 1,$$

so $x \in \ell^\infty$.

Thus $x \in \ell^p$ if and only if $p \in (2, \infty]$.

[4 marks]

A4. $\langle\cdot,\cdot\rangle : H \to \mathbb{C}$ is an inner product if, for all $x, y, z \in H$ and $\lambda, \mu \in \mathbb{C}$,

(1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;
(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
(3) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$.

[3 marks]

A5. The dual spce V^* is the space of all bounded linear functionals $f : V \to \mathbb{C}$. [Also OK to write "continuous" instead of "bounded".]

The norm $\|f\|$ of $f \in V^*$ is defined by [either formula OK]

$$\|f\| = \sup_{\substack{x\in V \\ x\neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in V \\ \|x\|=1}} |f(x)|.$$

[4 marks]

A6. An isometric isomorphism is a linear map $T : V \to V'$ between two vector spaces with norms $\|\cdot\|$ and $\|\cdot\|'$ such that

(1) T is a bijection;
(2) $\|T(x)\|' = \|x\|$, for all $x \in V$.

If H is a Hilbert space then $T : H \to H^* : x \mapsto f_x$, where $f_x(y) = \langle y, x\rangle$, is an isometric isomorphism.

[3 marks]

A7.(i) An eigenvalue ot T is a number $\lambda \in \mathbb{C}$ such that $Tx = \lambda x$, for some non-zero $x \in V$.

(ii) The spectrum of T is the set

$$\mathrm{spec}(T) = \{z \in \mathbb{C} : (zI - T) : V \to V \text{ is not invertible }\}.$$

[4 marks]

B8.(i) A normed vector space is a Banach space if it is complete (i.e. if every Cauchy sequence converges).

[2 marks]

(ii) Suppose that $\{(x_i^{(n)})_{i=1}^\infty\}_{n=1}^\infty$ is a Cauchy sequence for $\|\cdot\|_1$. Given $\epsilon > 0$, there exists $N \geq 1$ such that, for $n, m \geq N$,

$$\sum_{i=1}^\infty |x_i^{(n)} - x_i^{(m)}| = \|(x_i^{(n)})_{i=1}^\infty - (x_i^{(m)})_{i=1}^\infty\|_1 < \epsilon \qquad (*)$$

and so, in particular, for any i,

$$|x_i^{(n)} - x_i^{(m)}| < \epsilon.$$

Hence, for each $i \geq 1$, $\{x_i^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$ (complete) and so has a limit $x_i \in \mathbb{C}$.

For any $M \geq 1$ and $n, m \geq N$, we have

$$\sum_{i=1}^M |x_i^{(n)} - x_i^{(m)}| \leq \sum_{i=1}^\infty |x_i^{(n)} - x_i^{(m)}| < \epsilon.$$

Let $m \to +\infty$ to obtain

$$\sum_{i=1}^M |x_i^{(n)} - x_i| \leq \epsilon, \qquad (**)$$

for any $M \geq 1$ and $n \geq N$. Then

$$\begin{aligned}\sum_{i=1}^M |x_i| &\leq \sum_{i=1}^M |x_i^{(N)} - x_i| + \sum_{i=1}^M |x_i^{(N)}| \\ &\leq \epsilon + \|(x_i^{(N)})_{i=1}^\infty\|_1.\end{aligned}$$

and, letting $M \to +\infty$, $\sum_{i=1}^\infty |x_i|$ is finite, so $(x_i)_{i=1}^\infty \in l^1$.

Letting $M \to +\infty$ in $(**)$ gives

$$\sum_{i=1}^\infty |x_i^{(n)} - x_i| \leq \epsilon,$$

for all $n \geq N$, so that $\lim_{n\to+\infty} \|(x_i^{(n)})_{i=1}^\infty - (x_i)_{i=1}^\infty\|_1 = 0$, as required.

[14 marks]

(iii) Consider, for example, the sequence $x^{(n)} = (x_i^{(n)})_{i=1}^\infty \in \ell^1$ defined by

$$x^{(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right).$$

Suppose $n > m$. We have

$$\|x^{(n)} - x^{(m)}\|_2 = \sum_{i=m+1}^{n} \frac{1}{i^2} \to 0,$$

as $n, m \to +\infty$, since $\sum_{i=1}^{\infty} 1/i^2$ converges. Thus $x^{(n)}$ is a Cauchy sequence.

Let

$$x = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i}, \dots\right).$$

Then

$$\lim_{n\to+\infty} \|x^{(n)} - x\|_2 = \lim_{n\to+\infty} \sum_{i=n+1}^{\infty} \frac{1}{i^2} = 0,$$

again since $\sum_{i=1}^{\infty} 1/i^2$ converges, i.e., $x^{(n)}$ converges to x wrt $\|\cdot\|_2$ (and to no other point in ℓ^2). However

$$\sum_{i=1}^{\infty} \frac{1}{i} = +\infty,$$

so $x \notin \ell^1$. This shows that $(\ell^1, \|\cdot\|_2)$ is not complete and hence is not a Banach space.

[9 marks]

B9.(i) Since f is continuous at 0, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|z\| < \delta \quad \Longrightarrow \quad |f(z)| < \epsilon.$$

Now suppose that $x \in V$ and that $y \in V$ satisfies $\|y - x\| < \delta$. Then

$$|f(y) - f(x)| = |f(y - x)| < \epsilon,$$

so f is continuous at x.
[5 marks]

(ii) Linearity of f follows from standard properties of the integral.
For $\phi \in C([0,1], \mathbb{R})$, we have

$$|f(\phi)| = \left| \int_0^1 \sin(\pi x)\phi(x)dx \right| \leq \left| \int_0^1 \sin(\pi x)dx \right| \|\phi\|_\infty = \left| \left[\frac{-\cos(\pi x)}{\pi} \right]_0^1 \right| \|\phi\|_\infty = \frac{2}{\pi}\|\phi\|_\infty.$$

This shows that f is bounded and that

$$\|f\| \leq \frac{2}{\pi}.$$

Now take $\phi = 1$, so $\|\phi\|_\infty = 1$. Then

$$|f(1)| = \left| \int_0^1 \sin(\pi x)dx \right| = \frac{2}{\pi},$$

so

$$\|f\| = \sup_{\|\phi\|_\infty = 1} |f(\phi)| \geq \frac{2}{\pi}.$$

Thus

$$\|f\| = \frac{2}{\pi}.$$

[10 marks]

(iii) Note that $i/(i+1) \leq 1$ for all $i \geq 1$. Since, for $x \in \ell^1$,

$$|g(x)| = \left| \sum_{i=1}^\infty \frac{i}{i+1} x_i \right| \leq \sum_{i=1}^\infty \frac{i}{i+1} |x_i| \leq \sum_{i=1}^\infty |x_i| = \|x\|_1,$$

we see that g is well defined, bounded and $\|g\| \leq 1$. Linearity is immediate from the definition.

Note that $i/(i+1) \to 1$, as $i \to +\infty$. Given $\epsilon > 0$, choose j sufficiently large that

$$\frac{j}{j+1} > 1 - \epsilon.$$

Define $y \in \ell^1$ by

$$y_i = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

Then

$$\|y\|_1 = 1$$

and

$$|g(y)| = \left| \sum_{i=1}^{\infty} \frac{i}{i+1} y_i \right| = \frac{j}{j+1} > 1 - \epsilon,$$

so that

$$\|g\| \geq \sup_{\|x\|_1 = 1} |g(x)| > 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\|g\| \geq 1$$

and hence

$$\|g\| = 1.$$

[10 marks]

B10.(i) *Cauchy-Schwarz inequality:* For all $x, y \in H$,

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

[3 marks]

(ii) From the definition, $\|x\| = \langle x, x\rangle^{1/2} \geq 0$ and

$$\|x\| = 0 \quad \iff \quad \langle x, x\rangle - 0 \quad \iff \quad x = 0.$$

For $x \in H$, $\lambda \in \mathbb{C}$,

$$\|\lambda x\| = \langle \lambda x, \lambda x x\rangle^{1/2} = \left(\lambda\overline{\lambda}\langle x, x\rangle\right)^{1/2} = \left(|\lambda|^2\langle x, x\rangle\right)^{1/2} = |\lambda|\langle x, x\rangle^{1/2}.$$

For $x, y \in H$,

$$\begin{aligned}
\|x + y\| = \langle x + y, x + y\rangle &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \langle x, x\rangle + 2\mathrm{Re}(\langle x, y\rangle) + \langle y, y\rangle \\
&\leq \langle x, x\rangle + 2\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2} + \langle y, y\rangle \\
&= (\langle x, x\rangle^{1/2} + \langle y, y\rangle^{1/2})^2 = (\|x\| + \|y\|)^2,
\end{aligned}$$

using the Cauchy-Schwarz inequality to get from line 2 to line 3. Taking square roots gives the triangle inequality.

[7 marks]

(iii) The orthogonal complement $L^\perp$ is defined by

$$L^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in L\}.$$

[2 marks]

(iv)(a) L is clearly a subspace of ℓ^2 because if $(x_i), (y_i) \in \ell^2$ have $x_1 = 0$ and $y_1 = 0$ then $ax_1 + by_1 = 0$, for any $a, b \in \mathbb{C}$.

To see that L is closed, suppose that, for $n \geq 1$,

$$x^{(n)} = (0, x_2^{(n)}, x_3^{(n)}, \dots)$$

is a sequence in ℓ^2 which converges to $x = (x_1, x_2, x_3, \dots) \in \ell^2$. Given $\epsilon > 0$, we can find n sufficiently large that

$$|x_1|^2 + \sum_{i=2}^{\infty} |x_i^{(n)} - x_i|^2 = \|x^{(n)} - x\|_2^2 < \epsilon^2.$$

In particular, $|x_1| < \epsilon$ and so, since ϵ is arbitrary, $x_1 = 0$. Thus $x \in L$ and so L is closed.

[5 marks]

(b) In ℓ^2,

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

Thus

$$L^\perp = \left\{ x = (x_1, x_2, x_3, \dots) \in \ell^2 : \sum_{i=1}^{\infty} x_i \overline{y_i} \text{ for all } (0, y_2, y_3, \dots) \in L \right\}.$$

Let e_i denote the vector with 1 in the ith place and 0 elsewhere. For $i \geq 2$, $e_i \in L$. Thus, if $x \in L^\perp$ then

$$x_i = \langle x, e_i\rangle = 0 \text{ for all } i \geq 2,$$

so

$$L^\perp \subset \{(x_1, 0, 0, \dots) : x_1 \in \mathbb{C}\}.$$

Also, for $(0, y_2, y_3, \dots) \in L$,

$$\langle (x_1, 0, 0, \dots), (0, y_2, y_3, \dots)\rangle = 0,$$

so

$$\{(x_1, 0, 0, \dots) : x_1 \in \mathbb{C}\} \subset L^\perp.$$

Therefore

$$L^\perp = \{(x_1, 0, 0, \dots) : x_1 \in \mathbb{C}\}.$$

[6 marks]

(c) If $x = (x_1, x_2, \dots) \in \ell^2$ then we may write x uniquely as $x = y + z$, where

$$y = (0, x_2, x_3, \dots) \in L$$

and

$$z = (x_1, 0, 0, \dots) \in L^\perp.$$

This shows that $\ell^2 = L \oplus L^\perp$.

[2 marks]

B11.(i) The adjoint of $T : H \to H$ is the operator $T^* : H \to H$, where defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x, y \in H.$$

(Bookwork)

[3 marks]

(ii)(a) The linearity of T is clear.
For $x = (x_1, x_2, x_3, \dots) \in \ell^2$, we have

$$\|T(x)\|_2 = \left(\sum_{i=2}^{\infty} |x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} = \|x\|_2.$$

This shows that T is bounded and that

$$\|T\| \leq 1.$$

Now set $x = (0, x_2, x_3, \dots)$. Then

$$\|T\|\|x\|_2 \geq \|T(x)\|_2 = \left(\sum_{i=2}^{\infty} |x_i|^2\right)^{1/2} = \|x\|_2,$$

so $\|T\| \geq 1$. Thus

$$\|T\| = 1.$$

[6 marks]

(b) For $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ in ℓ^2, we have the equation

$$\begin{aligned}\langle Tx, y\rangle &= x_2\overline{y_1} + x_3\overline{y_2} + \cdots \\ &= x_1 \cdot 0 + x_2\overline{y_1} + x_3\overline{y_2} + \cdots \\ &= \langle x, T^*y\rangle.\end{aligned}$$

Thus

$$T^*(y) = (0, y_1, y_2, \dots).$$

[4 marks]

(iii)(a) For $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ in ℓ^2, we have the equation

$$\begin{aligned}\langle Sx, y\rangle &= \frac{x_2}{2}\overline{y_1} + \frac{x_3}{3}\overline{y_2} + \cdots \\ &= x_1 \cdot 0 + \frac{x_2}{2}\overline{y_1} + \frac{x_3}{3}\overline{y_2} + \cdots \\ &= \langle x, T^*y\rangle.\end{aligned}$$

Thus

$$T^*(y) = \left(0, \frac{y_1}{2}, \frac{y_2}{3}, \dots\right).$$

[5 marks]

(b) From the definition of S,

$$\begin{aligned}
&S^n(x_1, x_2, \dots, x_i, \dots) \\
&= \left(\frac{x_{n+1}}{n(n-1)\cdots 2\cdot 1}, \frac{x_{n+2}}{(n+1)n\cdots 3\cdot 2}, \dots, \frac{x_{n+i}}{(n+i-1)(n+i-2)\cdots(i+1)i}, \dots\right).
\end{aligned}$$

For $x = (x_1, x_2, x_3, \dots) \in \ell^2$, we have

$$\|S^n(x)\|_2 = \left(\sum_{i=1}^{\infty}\left|\frac{x_{n+i}}{(n+i-1)(n+i-2)\cdots(i+1)i}\right|^2\right)^{1/2} \le \frac{1}{n!}\left(\sum_{i=1}^{\infty}|x_{n+i}|^2\right)^{1/2} \le \frac{1}{n!}\|x\|_2,$$

so that

$$\|S^n\| \le \frac{1}{n!}.$$

Let e_n denote the vector with 1 in the nth place and 0 elsewhere. Then $\|e_n\|_2 = 1$ and

$$\|S^n(e_n)\|_2 = \frac{1}{n!},$$

so that $\|S^n\| \ge 1/n!$. Hence

$$\|S^n\| = 1/n!.$$

By the spectral radius formula, the spectral radius $\rho(S)$ of S is

$$\begin{aligned}
\rho(S) &= \lim_{n\to+\infty}\|S^n\|^{1/n} = \lim_{n\to+\infty}(1/n!)^{1/n} \\
&= \lim_{n\to+\infty}\left(\frac{n^{n+1/2}e^{-n}}{n!}\right)^{1/n}\left(\frac{1}{n^{n+1/2}e^{-n}}\right)^{1/n} = \lim_{n\to+\infty}\frac{1}{n}\frac{e}{n^{1/2n}} = 0,
\end{aligned}$$

using Stirling's formula.

[7 marks]

C12.(i) Suppose that

$$C = \{(x_i^{(n)})_{i=1}^{\infty} : n = 1, 2, 3, \dots\}$$

is a countable subset of $\ell^\infty(\mathbb{R})$. Define $(y_i)_{i=1}^\infty \in \ell^\infty(\mathbb{R})$ by

$$y_i = x_i^{(i)} - 1 \quad \forall\, i \geq 1.$$

Then, for each $n \geq 1$,

$$\|(x_i^{(n)})_{i=1}^\infty - (y_i)_{i=1}^\infty\|_\infty = \sup_{i \geq 1} |x_i^{(n)} - y_i| \geq |x_n^{(n)} - y_n| = 1.$$

Thus $(y_i)_{i=1}^\infty$ is distance at least 1 from each element of C, so C cannot be dense in $\ell^\infty(\mathbb{R})$. This shows that $\ell^\infty(\mathbb{R})$ is not separable.

[10 marks]

(ii) The space $c_0(\mathbb{R})$ *is* separable. To see this, consider, for each $N \geq 1$, the set

$$D_N = \{(x_i)_{i=1}^\infty \mid x_i \in \mathbb{Q},\ x_i = 0 \text{ for } i \geq N\}.$$

Clearly $D_N \subset c_0(\mathbb{R})$. Furthermore, each D_N is countable (since it is the Cartesian product of N copies of $\mathbb{Q}$) and therefore so is

$$D = \bigcup_{N=0}^{\infty} D_N \subset c_0(\mathbb{R}).$$

Now suppose that $(y_i)_{i=1}^\infty \in c_0(\mathbb{R})$ and choose $\epsilon > 0$. By the definition of $c_0(\mathbb{R})$, there exists $N \geq 0$ such that

$$i \geq N \quad \Longrightarrow \quad |y_i| < \epsilon.$$

Also, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $(x_i)_{i=1}^\infty \in D_N$ such that

$$|y_i - x_i| < \epsilon,\ i = 1, \dots, N-1.$$

Then

$$\|(y_i)_{i=1}^\infty - (x_i)_{i=1}^\infty\|_\infty = \sup\{|y_1 - x_1|, \dots, |x_{N-1} - y_{N-1}|, |y_N|, |y_{N+1}|, \dots\} \leq \epsilon.$$

Therefore D is dense in $c_0(\mathbb{R})$.

[15 marks]

(iii) Let $x = (x_i)_{i=1}^\infty \in \ell^1(\mathbb{R})$ and let $\epsilon > 0$. Since $\sum_{i=1}^\infty |x_i| < +\infty$, there exists $N \geq 1$ such that

$$\sum_{i=N+1}^{\infty} |x_i| < \epsilon.$$

Let

$$y = (x_1, x_2, \dots, x_N, 0, 0, \dots) = \sum_{n=1}^{N} x_n e_n.$$

Since this is a *finite* linear combination of the e_n's, we have

$$y \in \text{span}(\{e_n\}_{n=1}^{\infty}).$$

Furthermore,

$$\|x - y\|_1 = \sum_{i=1}^{\infty} |x_i - y_i| = \sum_{i=N+1}^{\infty} |x_i| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows that $\text{span}(\{e_n\}_{n=1}^{\infty})$ is dense in $\ell^1(\mathbb{R})$.

[10 marks]

(iv) See the proof of Lemma E.2 in the extra reading.

[20 marks]

(v) *Hahn-Banach Theorem:* Let V be a normed vector space over $\mathbb{R}$ and let $W \subset V$ be a linear subspace. Suppose that $f \in W^*$. Then f can be extended to a linear functional $\widetilde{f} \in V^*$ with $\|\widetilde{f}\| = \|f\|$.

[5 marks]