Lecture 11

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20912 - Introduction to Financial Mathematics
1. American Put Option Pricing on Binomial Tree

2. Replicating Portfolio
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\[ E - (P + S) > 0 \]
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  $$P^m_n = e^{-r\Delta t} \left( pP^m_{n+1} + (1 - p)P^m_{n+1} \right).$$

Here $0 \leq n \leq m$ and the risk-neutral probability $p = \frac{e^{r\Delta t} - d}{u - d}$. 
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- **American Put Option:**

  \[ P^m_n = \max \left\{ \max(E - S^m_n, 0), e^{-r\Delta t} \left( pP^m_{n+1} + (1 - p)P^m_{n+1} \right) \right\}, \]

  where $S^m_n$ is the $n$-th possible value of stock price at time-step $m\Delta t$. 
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  where $S_{n}^{m}$ is the $n$-th possible value of stock price at time-step $m\Delta t$.

- **Final condition:** $P_{n}^{N} = \max \left( E - S_{n}^{N}, 0 \right)$, where $n = 0, 1, 2, ..., N$, $E$ is the strike price.
We assume that over each of the next two years the stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.

Find the value of a 2-year American put with a strike price of $52 on a stock whose current price is $50.
Example: Evaluation of American Put Option on Two-Step Tree

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Find the value of a 2-year American put with a strike price of $52 on a stock whose current price is $50.

In this case $u = 1.2$, $d = 0.8$, $r = 0.05$, $E = 52$.

Risk-neutral probability: $p = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282$
Example: Evaluation of American Put Option on Two-Step Tree

\[ P_u = e^{-0.05 \times 1} (0.6282 \times 0 + 0.3718 \times 4) = 1.4147 \]
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Payoff: \( E - S = 52 - 40 = 12 > 9.4636 \). Early exercise is optimal!

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Payoff: \( E - S = 52 - 50 = 2 < 5.0894 \). Early exercise is not optimal at the initial node.
Replicating Portfolio

The aim is to calculate the value of call option $C_0$.

Let us establish a portfolio of stocks and bonds in such a way that the payoff of a call option is completely replicated.

Final value: $\Pi_T = C_T = \max (S - E, 0)$
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The Law of One Price: $\Pi_t = C_t$.

Consider replicating portfolio of $\Delta$ shares held long and $N$ bonds held short. The value of portfolio: $\Pi = \Delta S - NB$. A pair $(\Delta, N)$ is called a trading strategy.

How to find $(\Delta, N)$ such that $\Pi_T = C_T$ and $\Pi_0 = C_0$?
Example: One-Step Binomial Model.

Initial stock price is $S_0$. The stock price can either move up from $S_0$ to $S_0 u$ or down from $S_0$ to $S_0 d$. At time $T$, let the option price be $C_u$ if the stock price moves up, and $C_d$ if the stock price moves down.
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When stock moves up: \( \Delta S_0 u - NB_0 e^{rT} = C_u \).

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We have two equations for two unknown variables $\Delta$ and $N$. 
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We have two equations for two unknown variables $\Delta$ and $N$.

Current value: $C_0 = \Delta S_0 - NB_0$.

Prove: $C_0 = e^{-rT}(pC_u + (1 - p)C_d)$, where $p = \frac{e^{rT} - d}{u - d}$. (Exercise sheet 5)