MATH 20201 Algebraic Structures I. Exercises.

Sheet 1: Permutations

1. Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}, \ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}. \)

   (i) Work out \( \tau \circ \sigma, \sigma \circ \tau, \tau \circ \tau \) and \( \sigma \circ \sigma \) (in standard notation).

   (ii) Write \( \sigma, \tau, \tau \circ \sigma, \sigma \circ \tau, \tau \circ \tau \) and \( \sigma \circ \sigma \) as a composite of disjoint cycles.

2. Write the following permutations of \( \Omega = \{1, 2, 3, 4, 5\} \) in standard notation. (i) \((134) \circ (25), \) (ii) \((134) \circ (134) \circ (134), \) (iii) \((231) \circ (321), \) (iv) \((1324) \circ (531) \circ (24), \)

3. Which of the following equations are true? (i) \((12345) = (34512), \) (ii) \((12345) = (23415), \) (iii) \((54321) = (32154), \) (iv) \((54321) = (15423).\)

4. Write

   \[
   \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 8 & 5 & 6 & 1 & 3 & 7 & 2 \end{pmatrix}
   \]

   and

   \[
   \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 9 & 1 & 3 & 4 & 8 & 7 & 2 & 6 \end{pmatrix}
   \]

   as a composite of disjoint cycles.

5. Write the following permutations of \( S_5 \) as a composite of disjoint cycles:

   (i) \((1234) \circ (13) \circ (24) \circ (1432), \)

   (ii) \((12345) \circ (1342) \circ (15432),\)

   (iii) \((15) \circ (14) \circ (13) \circ (12), \)

6. Prove that every non-trivial permutation of \( \Omega = \{1, 2, 3, \ldots, n\} \) can be written as a composite of less than \( n \) transpositions.
Sheet 2: Groups

1. Which of the following sets are groups with respect to the binary operation given? Give proofs!

(i) The set of all 2x2 matrices of the form \[
\begin{pmatrix}
a & a \\
0 & 0 \\
\end{pmatrix}
\] where \(a \in \mathbb{R}, a \neq 0;\) under matrix multiplication.

(ii) The set of all 2x2 matrices of the form \[
\begin{pmatrix}
a & b \\
0 & 0 \\
\end{pmatrix}
\] where \(a, b \in \mathbb{R}, a \neq 0 \neq b,\) under matrix multiplication.

(iii) The set \(\mathbb{Z} \times \mathbb{Z} = \{(a, b); a, b \in \mathbb{Z}\}\) with multiplication

\[(a, b)(n, m) = (a + n, b + m);\]

\(a, b, n, m \in \mathbb{Z}.\)

(iv) The set \(\mathbb{Z} \times \mathbb{Z}\) with multiplication

\[(a, b)(n, m) = (a + n, (-1)^n b + m);\]

\(a, b, n, m \in \mathbb{Z}.\)

(v) The power set \(P(\Omega)\) of a non empty set \(\Omega\) under intersection of sets.

(vi) \(P(\Omega)\) as in (v) with the binary operation

\(AB = (A \cup B) \setminus (A \cap B),\)

\(A, B \subseteq \Omega.\) (You may take for granted that this operation is associative.)

2. For the set \(G\) and the binary operation \(*\) on \(G\) as given below, determine if \((G, *)\) is a group. Give reasons if your answer is NO!

(i) \(G = \mathbb{Q}\setminus \{0\}, \quad a * b = ab\) (multiplication of numbers),

(ii) \(G = \{1, -1\}, \quad a * b = ab\) (multiplication of numbers),

(iii) \(G = \mathbb{Z}, \quad a * b = ab\) (multiplication of numbers),

(iv) \(G\) is the set of even integers, \(a * b = a + b\) (addition of numbers),

(v) \(G = \{f, g\},\) where \(f\) and \(g\) are functions from \(\mathbb{Z}\) to \(\mathbb{Z}\) defined by \(f(n) = n,\ g(n) = -n \forall n \in \mathbb{Z},\) \(* = \circ\) (composition of functions),
(vi) $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$, $\ast = \text{matrix multiplication},$

(vii) $G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$, $\ast = \circ \ (\text{composition of permutations}),$

(viii) $G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$, $\ast = \circ \ (\text{composition of permutations}),$

(ix) $G = \{0, 1\}$, $a \ast b = ab \ (\text{multiplication of numbers}),$

(x) $G$ is the set of odd integers, $a \ast b = ab \ (\text{multiplication of numbers}).$

3. Prove that $\mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$ is a group under the binary operation $(a, b)(c, d) = (ac - bd, bc + ad),$

where $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}.$

4. Let $G = \{e, a, b, c\},$ where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

and let $\ast$ be matrix multiplication. Write out the multiplication table for $(G, \ast)$ and prove that it is a group.

5. Let $G = \mathbb{R} \setminus \{1\},$ and $a \ast b = a + b - ab.$ Prove that $(G, \ast)$ is a group.

6. Let $G = \{e, a, b, c\}$ be the set of permutations of $\Omega = \{1, 2, 3, 4\}$ where

$$e = \text{id}, \quad a = (12), \quad b = (34), \quad c = (12) \circ (34)$$

and let $\ast = \circ. \ (\text{composition of permutations}).$ Write out the multiplication table for $(G, \ast)$ and prove that it is a group.

7. Using cycle notation, make a list of all elements of the symmetric group $S_4.$

8. Make a list of all elements of the group $GL(2, \mathbb{Z}_2).$
Sheet 3: Subroups

1. For the following subsets $S$ of the given group $(G, *)$ determine whether or not $S$ is a subgroup. Give reasons if your answer is NO.

   (i) $G = \mathbb{C}$, $* = +$, $S = \{n + mi | n, m \in \mathbb{Z}\}$,
   (ii) $G = \mathbb{C}$, $* = +$, $S = \{3n + mi | n, m \in \mathbb{Z}\}$,
   (iii) $G = \mathbb{Q}^* = \mathbb{Q}\{0\}$, $* = \times$ (multiplication of numbers), $S = \{1, -1\}$,
   (iv) $G$ is the set of all permutations of $\Omega = \{1, 2, ..., n\}$, $* = \circ$, $S$ is the set of all transpositions,
   (v) $G = \mathbb{Z}_6$, $* = \oplus$, $S = \{0, 2, 4\}$,
   (vi) $G = \mathbb{Z}_6$, $* = \oplus$, $S = \{0, 1, 3\}$.

2. Let $n \geq 2$ and $A_n$ denote the subset of $S_n$ consisting of all permutations of $\{1, 2, ..., n\}$ that can be written as a composite of an even number of transpositions. Prove that $A_n$ is a subgroup of $S_n$.

3. Work out the orders of the following elements.

   (i) $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{\sqrt{3}}{2} + \frac{1}{2}i \in \mathbb{C}^*$
   (ii) $4, 15, 18, 33 \in \mathbb{Z}_{36}$
   (iii) $\sigma, \tau, \theta \in S_9$, where
   
   \[
   \sigma = \begin{pmatrix}
   1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
   2 & 3 & 1 & 5 & 6 & 7 & 4 & 9 & 8
   \end{pmatrix},
   \tau = \begin{pmatrix}
   1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
   9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
   \end{pmatrix},
   \theta = \begin{pmatrix}
   1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
   3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 1
   \end{pmatrix}
   \]
   (iv) $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{R})$. 

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4. What are the largest possible orders of elements in \( S_3, S_5 \) and \( S_7 \)?

5. (i) Compute \( \langle 5 \rangle \) in \( \mathbb{Z} \).

(ii) Compute \( \langle (1234) \rangle \) in \( S_4 \).

(iii) Compute \( \langle 1 \rangle, \langle 2 \rangle \) and \( \langle 3 \rangle \) in \( \mathbb{Z}_8 \).

(iv) Compute \( \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \) in \( GL(2, \mathbb{R}) \).

6. Prove that in the symmetric group \( S_n \),
\[
C((12\ldots n-1)) = \langle (12\ldots n-1) \rangle,
\]
i.e. the centralizer of the cycle \((12\ldots n-1)\) (of length \(n-1\)) in \( S_n \) coincides with the cyclic subgroup generated by that cycle.

7. Using inspection of all the 24 elements of \( S_4 \) (or otherwise), work out the centralizer of the transposition \((12)\) in \( S_4 \).

8. Prove that the centralizer of the transposition \((12)\) in \( S_n \) with \( n \geq 4 \) is the subgroup \( C((12)) = \{ \sigma \in S_n \mid \sigma (12) = (12) \} \).

9. Work out the centralizer \( C(g) \) in \( GL(2, \mathbb{R}) \) in each of the following cases:

(i) \( g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \), (ii) \( g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), (iii) \( g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

10. Work out the centre of
\[
T(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} ; \ a, b, c \in \mathbb{R}, ac \neq 0 \right\}.
\]

11. Work out the centre of
\[
UT(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; \ a, b, c \in \mathbb{R} \right\}.
\]

12. Let \( G \) be a group and \( g \in G \).

(i) If \( C(g) = G \), what can we say about \( g \)?

(ii) If \( C(g) = G \) for all \( g \in G \), what can we say about \( G \)?
Sheet 4: Cyclic Groups

1. Which elements of the cyclic group \((\mathbb{Z}_{30}, \oplus)\) are generators?

2. Prove that \((\mathbb{Q}, +)\) is not a cyclic group.

3. Find all orders of subgroups of (i) \(\mathbb{Z}_{31}\), (ii) \(\mathbb{Z}_{32}\), (iii) \(\mathbb{Z}_{33}\).

4. Find all subgroups of \(\mathbb{Z}_{15}\).

5. If \(G\) is a cyclic group of order \(n\), and \(m\) divides \(n\), show that \(G\) contains a subgroup of order \(m\).
Sheet 5: Cosets, Lagrange’s Theorem

1. For each pair \(G, H\) (where \(H \leq G\)) determine \([G : H]\) and list all right cosets of \(H\) in \(G\).
   (i) \(G = \mathbb{Z}_{15}, H = \langle 12 \rangle,\)
   (ii) \(G = \{e, (12), (34), (12)(34)\}\) (a subgroup of the symmetric group \(S_4\)), \(H = \langle (12)(34) \rangle,\)
   (iii) \(G = \mathbb{R}^* = (\mathbb{R} \setminus \{0\}, \times), H = (\mathbb{R}_+, \times).\)

2. What are the right cosets of \(H = \{z \in \mathbb{C} \setminus \{0\}; |z| = 1\}\) in \(\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \times)\)?

3. For \(H \leq G\) as specified below, determine the right cosets of \(H\) in \(G\).
   (i) \(G = \mathbb{R}^*, H = \langle -1 \rangle\)
   (ii) \(G = \mathbb{C}^*, H = \mathbb{R}^*\)
   (iii) \(G = \mathbb{C}^*, H = \mathbb{R}_+\)
   (iv) \(G = \mathbb{Z}_{36}, H = \langle 30 \rangle\)
   (v) \(G = T(2, \mathbb{R}), H = UT(2, \mathbb{R})\)
   (vi) \(G = D(2, \mathbb{R}), H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \lambda \in \mathbb{R}, \lambda \neq 0 \right\}\)
   (vii) \(G = GL(n, \mathbb{R}), H = SL(n, \mathbb{R})\) [Hint: Show that the right coset determined by a matrix \(A \in GL(n, \mathbb{R})\) with determinant \(\det A = a\) is the set of all \(n \times n\) matrices \(B\) with determinant \(\det B = a.\)]

4. For \(G\) and \(H\) as in Ex.3, work out \([G : H]\), the index of \(H\) in \(G\).

5. Let \(G\) be a group, \(H \leq G,\) and \(x, y \in G\). Prove that \(Hx = Hy\) if and only if \(xy^{-1} \in H\).

6. Prove that the right cosets of \(\mathbb{Z}\) in \(\mathbb{R}\) are in 1-1 correspondence with the set \([0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}.\)
7. In the symmetric group $S_4$, work out the cosets
   (i) $H(134)$, (ii) $H(23)$, (iii) $H(1432)$,
   where $H = \langle (1234) \rangle$ is the cyclic subgroup generated by $(1234)$.


9. Let $G$ be a finite group with $|G| = n$. Prove that $g^n = e$ for all $g \in G$.

10. Let $G$ be a finite group and $K, H \leq G$ with $(|K|, |H|) = 1$, i.e. the orders of the subgroups $K$ and $H$ are coprime. Prove that $H \cap K = \{e\}$.
Sheet 6: Homomorphisms and Isomorphisms

1. Which of the following maps are group homomorphisms?

(a) $\varphi : \mathbb{C}^* \rightarrow \mathbb{R}$, $\varphi (z) = \log |z|$ ($z \in \mathbb{C}^*$)

(b) $\varphi : \mathbb{C} \rightarrow \mathbb{R}$, $\varphi (a + bi) = b$ ($a, b \in \mathbb{R}$)

(c) $\varphi : GL(2, \mathbb{R}) \rightarrow \mathbb{R}$, $\varphi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a - d$, $a, b, c, d \in \mathbb{R}$.

(d) $\varphi : T(2, \mathbb{R}) \rightarrow D(2, \mathbb{R})$, $\varphi \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a, b, d \in \mathbb{R}$, $ad \neq 0$

(e) $\varphi : S_n \rightarrow S_{n+1}$, where for $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1\sigma & 2\sigma & \cdots & n\sigma \end{pmatrix} \in S_n$,

$$\varphi (\sigma) = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ 1\sigma & 2\sigma & \cdots & n\sigma & n+1 \end{pmatrix} \in S_{n+1}$$

(f) $\varphi : \mathbb{Z} \rightarrow S_n$, $\varphi (k) = (123\ldots n)^k \in S_n$, $k \in \mathbb{Z}$

2. Let $\varphi : G \rightarrow H$ be a homomorphism of groups. Show that if $a \in G$ has order $n$, then $\varphi (a) \in H$ has order dividing $n$.

3. Let $G$ be a group. Show that the map $\varphi : G \rightarrow G$ defined by $\varphi (a) = a^{-1}$ is a homomorphism if and only if $G$ is abelian.

4. Let $G$ be a group and $x \in G$. Show that the map $\varphi_x : G \rightarrow G$ defined by $\varphi_x (a) = x^{-1}ax$ for all $a \in G$ (with $x$ fixed) is an isomorphism.

5. By using group-theoretic properties, show that the following statements are true.

(a) $S_4 \not\cong \mathbb{Z}_{24}$.

(b) $\mathbb{Z} \not\cong \mathbb{Q}$.

(c) $\mathbb{Q} \not\cong \mathbb{Q}^*$.

(d) $GL(2, \mathbb{R}) \not\cong UT(3, \mathbb{R})$.  

1. For $\sigma, \tau \in S_6$ as specified below, decide whether or not $\tau$ is a conjugate of $\sigma$. If your answer is "yes", find an element $\theta \in S_6$ such that $\tau = \theta^{-1}\sigma\theta$.

(a) $\sigma = (1234) (56)$, $\tau = (12) (3456)$,
(b) $\sigma = (12345)$, $\tau = (123456)$,
(c) $\sigma = (45) (36) (12)$, $\tau = (16) (25) (34)$,
(d) $\sigma = (12)(346)$, $\tau = (123) (56)$.

2. Work out the number of conjugacy classes in $S_5$ and $S_6$.

3. Determine the number of cycles of length $n$ in $S_n$. [Hint: Use Theorem 7.2, the class formula, and the fact that $C((12...n)) = \langle (12...n) \rangle$. This will be nicer than a purely combinatorial solution.]

4. Work out the order of the centralizer of $(12)(34)$ in $S_4$ and find all its elements. [Hint: Apart from obvious members of the centralizer, consider $(1324)$.]

5. Use the formula for conjugates in $S_n$ to prove that $Z(S_n) = \{e\}$ for $n \geq 3$.

6. Show that in any group conjugate elements have the same order.
Sheet 8 : Normal Subgroups

1. For a group $G$ and a subgroup $H \leq G$ as specified below, decide if $H$ is normal in $G$.
   
   (a) $G = S_4$, $H = \langle (1234) \rangle$;
   
   (b) $G = T(2, \mathbb{R})$, $H = UT(2, \mathbb{R})$;
   
   (c) $G = GL(2, \mathbb{R})$, $H =$ the subgroup of scalar matrices in $GL(2, \mathbb{R})$;
   
   (d) $G = S_4$, $H = \{e, (12) (34), (13) (24), (14) (23)\}$;
   
   (e) $G = SL(2, \mathbb{R})$, $H = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$.

2. Suppose $H \triangleleft G$ and $K \triangleleft G$. Show that $H \cap K \triangleleft G$.

3. Suppose that $H \triangleleft G$ and $K \triangleleft G$. The product of $H$ and $K$ in $G$ is the set $HK = \{hk : h \in H, k \in K\}$. Show that $HK \triangleleft G$.

4. Let $H \leq K \leq G$.
   
   (a) Suppose $H \triangleleft G$. Show that $H \triangleleft K$.
   
   (b) Suppose that $H \triangleleft K$ and $K \triangleleft G$. Does this imply that $H \triangleleft G$?

5. Let $N$ and $M$ be normal subgroups of $G$ such that $N \cap M = \{e\}$. Show that for all $x \in N$ and all $y \in M$ one has $xy = yx$. [Hint: Show that $x^{-1}y^{-1}xy = e$.]

6. Let $G$ be a group, and let $H \triangleleft G$ be a cyclic group of order two which is normal in $G$. Prove that $H \subseteq Z(G)$.
Sheet 9: Factor Groups and the First Isomorphism Theorem

1. For the maps in Ex. 6.1, if they are homomorphisms, find the kernel and the image.

2. Show that $\mathbb{R}^*/\mathbb{R}^+_+ \cong \{\pm 1\}$, where $\mathbb{R}^+_+ = \{r \in \mathbb{R}^*; r > 0\}$ and $\{\pm 1\}$ is regarded as a group under multiplication.

3. Show that $\mathbb{C}^*/\{z \in \mathbb{C}; |z| = 1\} \cong \mathbb{R}^+_+$, where $\mathbb{R}^+_+ = \{r \in \mathbb{R}^*; r > 0\}$.

4. Show that $T(2, \mathbb{R})/UT(2, \mathbb{R}) \cong D(2, \mathbb{R})$.

5. Let
   \[ N = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a, b \in \mathbb{R}, a \neq 0 \right\} \leq T(2, \mathbb{R}). \]
   Show that $N$ is normal in $T(2, \mathbb{R})$, and prove that $T(2, \mathbb{R})/N \cong \mathbb{R}^*$.
   [Hint: Consider the map $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \rightarrow ac$]

6. Let
   \[ N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; a, b \in \mathbb{R} \right\} \leq UT(3, \mathbb{R}). \]
   Show that $N$ is normal in $UT(3, \mathbb{R})$, and prove that $UT(3, \mathbb{R})/N \cong \mathbb{R}$.
   [Hint: Consider the map $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \rightarrow c$.]