Monte Carlo Extensions

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Outline

1. Review

Extension to Monte Carlo

Improving Monte Carlo

Simple extensions

Pseudo-Random Numbers

Early Exercise

Summary
1 REVIEW

2 EXTENSION TO MONTE CARLO
   - Improving Monte Carlo
   - Simple extensions
   - Pseudo-Random Numbers
   - Early Exercise
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2 Extension to Monte Carlo
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3 Summary
Simple to program and to understand
Convergence is slow, extrapolation impossible.
Forward looking method ideal for path dependent derivatives
Good for derivatives where there are multiple sources of uncertainty, as the computational effort only increases linearly.
To simulate the paths we typically use the solution to the SDE or the Euler approximation, along with a decent generator of Normally distributed random variables.
Overview

Here we will extend our basic theory and concentrate on some simple techniques to improve the basic method.

The techniques we will look at are:

- antithetic variables
- control variate technique
- moment matching importance sampling
- low discrepancy sequences.
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- antithetic variables
- control variate technique
- moment matching importance sampling
- low discrepancy sequences.

Most of these will reduce the variance of the error.

Low discrepancy sequences can also improve the convergence rate.
HOW TO IMPROVE IT

- Monte Carlo is typically the simplest numerical scheme to implement.
- But as you will see, its accuracy and uncertain convergence is not ideal for accurate valuation.
- However, it is often the only method available for complex problems.
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- but as you will see its accuracy and uncertain convergence is not ideal for accurate valuation.
- However, it is often the only method available for complex problems
- So we must be able to improve the accuracy of the standard model.
- We also will need a method to handle multiple Brownian motions AND early exercise
Antithetic variables or antithetic sampling is a simple adjustment to generating the $\phi_n (1 \neq n \leq N)$.

Instead of making $N$ independent draws, you draw the sample in pairs:

- if the $i$th Normally distributed variable is $\phi_i$;
- choose $\phi_{i+1}$ to be $-\phi_i$;
- then draw again for $\phi_{i+2}$.
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This guarantees that the mean of the two draws is zero.

We only need to make half as many draws for the same number of paths.

This should improve convergence.
Control variate technique: This is explained through an example:

- We want to compute $E^Q[V(T)]$
- And we can write

$$V(T) = V(T) - V_1(T) + V_1(T),$$

where

- $E^Q[V_1(T)]$ is known analytically
- and error in estimating $E^Q[V(T) - V_1(T)]$ by simulation is less than error in estimating $E^Q[V(T)]$
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Then, a better estimate of $E^Q[V(T)]$ is the sum of

- The known value of $E^Q[V_1(T)]$
- Plus the estimate of $E^Q[V(T) - V_1(T)]$
Moment matching is a simple extension to antithetic variables.

Using antithetic variables already matches mean and skewness.

To match all moments we must also match the variance of the required distribution.

We must match Brownian motion variance so set the variance of $\varphi$ to 1.

How to do this?

Take $N\varphi$ values and calculate their variance $v$.

Multiply all of the $\varphi$ values by $v - \frac{1}{2}$.

The variance of the new random draws is 1, as required.
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IMPORTANCE SAMPLING

- Importance sampling basically means choose only important paths (those which add value).
- In option pricing, that may be the $[a, b]$ region in $S_T$ where the payoff function has positive value.
- Sample from the distribution that cause $S_T$ to lie in $[a, b]$...
- multiply by the probability of $S_T$ being in this region.
- There exists a function that maps $[0, 1]$ onto $S_T$
- Invert to find $[x_1, x_2]$ that is mapped onto $[a, b]$
- To compute:
  - draw variables from $[0, 1]$, multiply by $x_2 - x_1$ and add $x_1$;
  - convert the $x$ value into $\phi$ and hence $S_T$;
  - determine the option value $V_T$ from $S_T$ and average values;
  - multiply this expectation by $(x_2 - x_1)$. 
Another method is Low discrepancy sequences (also known as Quasi Monte Carlo methods).

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LOW DISCREPANCY SEQUENCES

- Another method is Low discrepancy sequences (also known as Quasi Monte Carlo methods).
- In Monte Carlo we assume that with enough sample paths they will eventually cover the entire distribution.
- If we only draw a finite number of paths, truly 'random' numbers may cluster around particular values.
- This is not good for integration!
- To overcome this problem we throw away the idea of using 'random' numbers at all.
- Choose instead a deterministic sequence of numbers that does a very good job of covering the $[0, 1]$ interval.
- This method can improve the convergence of the Monte Carlo method from $1/N^2$ to $1/N$. 
AMERICAN OPTIONS

- One of the challenging areas in finance is how to value options with early exercise using Monte Carlo methods.
- Recall that the American option value $V_0$ is
  \[ V_0 = \max_{\tau} \mathbb{E}_\tau^Q[e^{-r\tau} \max(S_{\tau} - X, 0)] \]
- The problem comes from the fact that Monte Carlo is a forward looking method.
- To use the sample paths we would have to test early exercise at each point in time on each sample path.
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- The problem comes from the fact that Monte Carlo is a forward looking method.
- To use the sample paths we would have to test early exercise at each point in time on each sample path.
- How can we do this if we don’t know the option value at that point?
We have looked through a variety of extensions to the standard Monte Carlo in an effort to reduce the variance of the error or to improve the convergence.

Most of the improvements are simple to apply such as antithetic variables and moment matching, others are more complex such as low discrepancy sequences.

Finally, we looked at some of the early attempts to use Monte Carlo methods to value American style options. This is a precursor to the Longstaff and Schwartz approach.