FINITE DIFFERENCE METHODS

Dr P. V. Johnson

School of Mathematics

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Outline

1. Review
   - Last time
   - Today's Lecture

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   - Finite-difference approximations
   - Constructing the grid
   - Discretised equations

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2. Discretising the Problem
   - Finite-difference approximations
   - Constructing the grid
   - Discretised equations
3. Explicit finite difference method
   - System of equations
   - Stability and Convergence

Outline

- Analysed the binomial pricing model in detail including convergence rates
- Convergence is often non-monotonic due to nonlinearity error caused by discontinuities in the option price.
- There are methods of overcoming this, and it is particularly important for American options where there is no analytic solution.

- We now introduce the final numerical scheme which is related to the PDE solution.
- Finite difference methods are numerical solutions to (in CF, generally) parabolic PDEs.
- They work by
  - generating a discrete approximation to the PDE
  - solving the resulting system of the equations.
- There are three types of methods:
  - the explicit method, (like the trinomial tree),
  - the implicit method (best stability)
  - the Crank-Nicolson method (best convergence characteristics).

Approximations

- Consider a function of two variables $V(S,t)$, if we consider small changes in S and t we can use a Taylor's series to express $V(S+\Delta S,t)$, $V(S-\Delta S,t)$, $V(S,t+\Delta t)$ as follows (all the derivatives are evaluated at (S,t))

$$V(S+\Delta S,t) = V(S,t) + \Delta S\frac{\partial V}{\partial S} + \tfrac{1}{2}(\Delta S)^2\frac{\partial^2 V}{\partial S^2} + O((\Delta S)^3)$$

$$V(S-\Delta S,t) = V(S,t) - \Delta S\frac{\partial V}{\partial S} + \tfrac{1}{2}(\Delta S)^2\frac{\partial^2 V}{\partial S^2} + O((\Delta S)^3)$$

$$V(S,t+\Delta t) = V(S,t) + \Delta t\frac{\partial V}{\partial t} + \tfrac{1}{2}(\Delta t)^2\frac{\partial^2 V}{\partial t^2} + O((\Delta t)^2)$$

- In order to use a finite difference scheme we need to approximate derivatives
- For S, we have two options for the first derivative:
  - From (1) (or (2)) equation:

$$\begin{aligned}\frac{\partial V}{\partial S}(S,t) &= \frac{V(S+\Delta S,t)-V(S,t)}{\Delta S} + \frac{1}{2}\Delta S\frac{\partial^2 V}{\partial S^2} + O((\Delta S)^2) \\ &= \frac{V(S+\Delta S,t)-V(S,t)}{\Delta S} + O(\Delta S)\end{aligned}$$

  - From equations (1) and (2):

$$\frac{\partial V}{\partial S}(S,t) = \frac{V(S+\Delta S,t)-V(S-\Delta S,t)}{2\Delta S} + O((\Delta S)^2)$$

- For the second derivative we use equations (1) and (2)

$$\frac{\partial^2 V}{\partial S^2}(S,t) = \frac{V(S+\Delta S,t) - 2V(S,t) + V(S-\Delta S,t)}{(\Delta S)^2} + O((\Delta S)^2)$$

- For t we have

$$\begin{aligned}\frac{\partial V}{\partial t}(S,t) &= \frac{V(S,t+\Delta t) - V(S,t)}{\Delta t} + \frac{1}{2}\Delta t\frac{\partial^2 V}{\partial t^2} + O((\Delta t)^2) \\ &= \frac{V(S,t+\Delta t) - V(S,t)}{\Delta t} + O(\Delta t)\end{aligned}$$

How does this help us?

- Reconsider the Black-Scholes equation and in particular the Black-Scholes equation for a European options where there are continuous dividends:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0$$

- The boundary conditions for a call are:

$$V(S,T) = \max(S-X,0)$$

$$V(0,t) = 0$$

$$V(S,t) \to Se^{-\delta(T-t)} - Xe^{-r(T-t)} \quad \text{as} \quad S \to \infty$$

- and boundary conditions for a put are:

$$V(S,T) = max(X - S, 0)$$

$$V(0,t) = Xe^{-r(T-t)}$$

$$V(S,t) \to 0 \quad \text{as} \quad S \to \infty$$

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- We will now form a finite difference grid that describes the $S - t$ space in which we need to solve the Black-Scholes equation.
- For a numerical method we need to truncate the range of S.

- We now need to ensure that we have a fine enough grid to allow for most possible movements in S and enough time steps t.
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- As for the binomial and Monte-Carlo method we will discuss later what is a suitable size/number for these steps.
- Partition the interval $[0, S^U]$ into *jmax* subintervals each of length $\Delta S = S^U / jmax$.
- Partition the interval $[0, T]$ into *imax* subintervals each of length $\Delta t = T / imax$.
- We will denote the value at each node $V(j\Delta S, i\Delta t)$ as V_j^i

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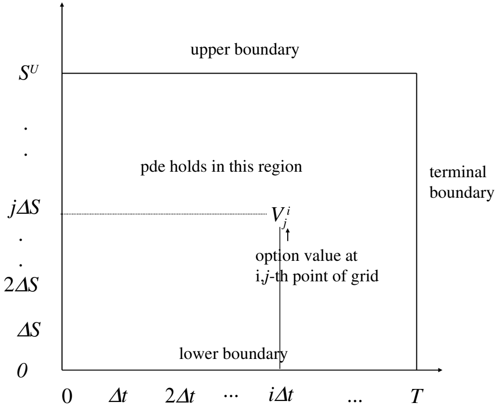
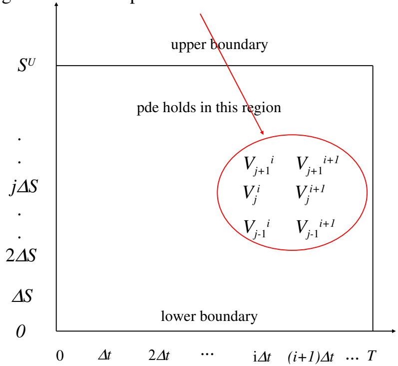
Finite difference grid



Finite difference grid

Focus attention on i, j-th value V_j^i, and a little piece of the grid around that point



- We clearly know the information at $t = T$ as this is the payoff from the option, by limiting our focus on

$$\begin{array}{cc} & V_{j+1}^{i+1} \\ V_j^i & V_j^{i+1} \\ & V_{j-1}^{i+1} \end{array}$$

- we can approximate the derivatives in the Black-Scholes equation by using our difference equations
- from this we can write V_j^i in terms of the other three terms.

- Recall the BSM equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0$$

- The BSM equation approximates to

$$\frac{V_j^{i+1} - V_j^i}{\Delta t} + \tfrac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} - V_{j-1}^{i+1}}{(\Delta S)^2} + (r-\delta) j\Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_j^i = 0$$

the unknown here is V_j^i as we have been working backward in time.

- The discretised BSM equation is

$$\frac{V_j^{i+1} - V_j^i}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{(\Delta S)^2}$$
$$+(r-\delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_j^i = 0$$

- Need to find V_j^i so rearrange in terms of this unknown:

$$V_j^i = \frac{1}{1 + r\Delta t}\left(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1}\right) \quad (*)$$

where:

$$A = \left(\tfrac{1}{2}\sigma^2 j^2 + \tfrac{1}{2}(r-\delta)j\right)\Delta t$$
$$B = 1 - \sigma^2 j^2 \Delta t$$
$$C = \left(\tfrac{1}{2}\sigma^2 j^2 - \tfrac{1}{2}(r-\delta)j\right)\Delta t$$

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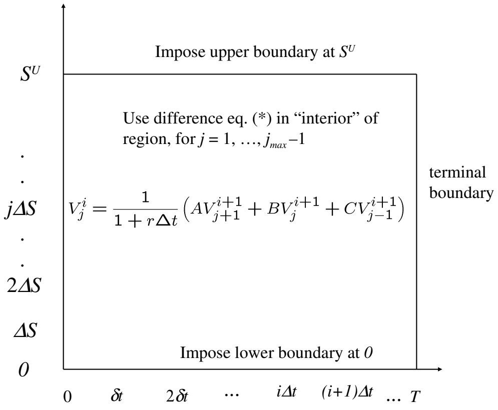
  - we have a way of calculating the option value at expiry
  - and we have scheme for calculating the option value at the previous time step.
- The differences between the binomial and explicit finite difference method are
  - the binomial uses two nodes to the explicit finite difference's three.
  - You get to choose the specifications of the grid in the finite difference method
  - You also need to specify the behaviour on the upper and lower S boundaries.

The grid again:



Boundary Conditions

- If we attempt to use equation (*) to calculate V_0^i then we need to have values of V_{-1}^i which we don't have (e.g. for calls):
- So for V_0^i and V_{jmax}^i we need to use our boundary conditions.
  - $V_0^i = 0$
  - $V_{jmax}^i = S^u e^{-\delta(T - i\Delta t)} - X e^{-r(T - i\Delta t)}$
- These conditions will naturally be different for different options, such as barrier options, put options etc.

PROBABILISTIC INTERPRETATION

- The explicit finite difference scheme is like a trinomial tree.
- Note that $A + B + C = 1$.
- We can also show that the expected value of S is at time $i\Delta t$:

$$E[S_j^i] = \frac{1}{1 + r\Delta t} E[S_j^{i+1}] \tag{1}$$

the expected future value of S, following GBM, under the risk-neutral probability discounted at the risk-free rate.

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  the expected future value of S, following GBM, under the risk-neutral probability discounted at the risk-free rate.
- A, B and C can then be interpreted as the risk-neutral probabilities.

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- Probabilistic ideas can be used to derive conditions for stability
- If we consider A, B and C as probabilities, we require that $A, B, C \geq 0$.
- For A and C this requires:

$$j > \left|\frac{r-\delta}{\sigma^2}\right|$$

- A far bigger problem is for B where this says that

$$\Delta t < \frac{1}{\sigma^2 j^2}$$

which means that you need to ensure that the time interval is small enough.

- The stability therefore restricts your choice of Δt, ΔS
  - Δt cannot be too small, or else computation will take too long
  - then this puts lower bound on size of ΔS

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CONVERGENCE

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- The errors will arise from only approximating the derivatives, in particular, in the explicit finite difference method:

$$\frac{\partial^2 V}{\partial S^2}(S,t) = \frac{V(S+\Delta S,t) - 2V(S,t) + V(S-\Delta S,t)}{(\Delta S)^2} + O((\Delta S)^2)$$

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- Further analysis shows that the errors decrease linearly in time steps
- and quadratic in steps in S.

Nonlinearity error

- Theoretical convergence rates depends upon all of the derivatives being well behaved (e.g. not infinite).
- However, we know that in the case of European options, the payoff at expiry is discontinuous leading to an infinite first derivative - and so it seems likely that our approximation may not work as well here.

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- There are therefore problems with any option that introduces a new boundary

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Summary

- Introduced the finite-difference method to solve PDEs
- Discretise the original PDE to obtain a linear system of equations to solve.
- This scheme was explained for the Black Scholes PDE and in particular we derived the explicit finite difference scheme to solve the European call and put option problems.

Summary

- Introduced the finite-difference method to solve PDEs
- Discretise the original PDE to obtain a linear system of equations to solve.
- This scheme was explained for the Black Scholes PDE and in particular we derived the explicit finite difference scheme to solve the European call and put option problems.
- The convergence of the method is similar to the binomial tree and, in fact, the method can be considered a trinomial tree.
- Explicit method can be unstable - constraints on our grid size.