Finite Difference Methods

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- Last time
- Today’s Lecture
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2 **Discretising the Problem**
   - Finite-difference approximations
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   - Discretised equations
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3 EXPLICIT FINITE DIFFERENCE METHOD
- System of equations
- Stability and Convergence
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4 OVERVIEW
- Summary
Analysed the binomial pricing model in detail including convergence rates.

Convergence is often non-monotonic due to nonlinearity error caused by discontinuities in the option price.

There are methods of overcoming this, and it is particularly important for American options where there is no analytic solution.
We now introduce the final numerical scheme which is related to the PDE solution.

Finite difference methods are numerical solutions to (in CF, generally) parabolic PDEs.

They work by
- generating a discrete approximation to the PDE
- solving the resulting system of the equations.

There are three types of methods:
- the explicit method, (like the trinomial tree),
- the implicit method (best stability)
- the Crank-Nicolson method (best convergence characteristics).
Consider a function of two variables $V(S, t)$, if we consider small changes in $S$ and $t$ we can use a Taylor’s series to express $V(S + \Delta S, t)$, $V(S - \Delta S, t)$, $V(S, t + \Delta t)$ as follows (all the derivatives are evaluated at $(S, t)$)

\[
V(S + \Delta S, t) = V(S, t) + \Delta S \frac{\partial V}{\partial S} + \frac{1}{2} (\Delta S)^2 \frac{\partial^2 V}{\partial S^2} + O((\Delta S)^3)
\]

\[
V(S - \Delta S, t) = V(S, t) - \Delta S \frac{\partial V}{\partial S} + \frac{1}{2} (\Delta S)^2 \frac{\partial^2 V}{\partial S^2} + O((\Delta S)^3)
\]

\[
V(S, t + \Delta t) = V(S, t) + \Delta t \frac{\partial V}{\partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 V}{\partial t^2} + O((\Delta t)^2)
\]
In order to use a finite difference scheme we need to approximate derivatives.

For $S$, we have two options for the first derivative:

- From (1) (or (2)) equation:

$$
\frac{\partial V}{\partial S}(S, t) = \frac{V(S + \Delta S, t) - V(S, t)}{\Delta S} + \frac{1}{2} \Delta S \frac{\partial^2 V}{\partial S^2} + O((\Delta S)^2)
$$

$$
= \frac{V(S + \Delta S, t) - V(S, t)}{\Delta S} + O(\Delta S)
$$

- From equations (1) and (2):

$$
\frac{\partial V}{\partial S}(S, t) = \frac{V(S + \Delta S, t) - V(S - \Delta S, t)}{2\Delta S} + O((\Delta S)^2)
$$
For the second derivative we use equations (1) and (2)

\[
\frac{\partial^2 V}{\partial S^2}(S, t) = \frac{V(S + \Delta S, t) - 2V(S, t) + V(S - \Delta S, t)}{(\Delta S)^2} + O((\Delta S)^2)
\]

For \( t \) we have

\[
\frac{\partial V}{\partial t}(S, t) = \frac{V(S, t + \Delta t) - V(S, t)}{\Delta t} + \frac{1}{2} \Delta t \frac{\partial^2 V}{\partial t^2} + O((\Delta t)^2)
\]

\[
= \frac{V(S, t + \Delta t) - V(S, t)}{\Delta t} + O(\Delta t)
\]
Reconsider the Black-Scholes equation and in particular the Black-Scholes equation for a European options where there are continuous dividends:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0
\]

The boundary conditions for a call are:

\[
V(S, T) = \max(S - X, 0)
\]

\[
V(0, t) = 0
\]

\[
V(S, t) \rightarrow S e^{-\delta(T-t)} - X e^{-r(T-t)} \quad \text{as} \quad S \rightarrow \infty
\]
and boundary conditions for a put are:

\[ V(S, T) = \max(X - S, 0) \]

\[ V(0, t) = Xe^{-r(T-t)} \]

\[ V(S, t) \to 0 \quad \text{as} \quad S \to \infty \]

We will now form a finite difference grid that describes the \( S - t \) space in which we need to solve the Black-Scholes equation.
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We will now form a finite difference grid that describes the \( S - t \) space in which we need to solve the Black-Scholes equation.

For a numerical method we need to truncate the range of \( S \).
We now need to ensure that we have a fine enough grid to allow for most possible movements in $S$ and enough time steps $t$.

As for the binomial and Monte-Carlo method we will discuss later what is a suitable size/number for these steps.
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As for the binomial and Monte-Carlo method we will discuss later what is a suitable size/number for these steps.

Partition the interval $[0, S^U]$ into $j_{max}$ subintervals each of length $\Delta S = S^U / j_{max}$.

Partition the interval $[0, T]$ into $i_{max}$ subintervals each of length $\Delta t = T / i_{max}$.

We will denote the value at each node $V(j\Delta S, i\Delta t)$ as $V^i_j$. 
Discretising the Problem

Explicit finite difference method

Overview

Finite-difference approximations

Constructing the grid

Discretised equations

Finite difference grid

0 2Δt ... iΔt ...

upper boundary

lower boundary

terminal boundary

$pde$ holds in this region

option value at $i,j$-th point of grid

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Finite difference grid

Focus attention on \( i, j \)-th value \( V_{j}^{i} \), and a little piece of the grid around that point.
We clearly know the information at $t = T$ as this is the payoff from the option, by limiting our focus on

$$
\begin{align*}
V_{i+1}^{j+1} \\
V_j^i \\
V_j^{i+1} \\
V_{j-1}^{i+1}
\end{align*}
$$

we can approximate the derivatives in the Black-Scholes equation by using our difference equations

from this we can write $V_j^i$ in terms of the other three terms.
Recall the BSM equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0 \]

The BSM equation approximates to

\[ \frac{V_{j+1}^i - V_j^i}{\Delta t} + \frac{1}{2} \sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^i - 2V_j^i - V_{j-1}^i}{(\Delta S)^2} \]

\[ +(r - \delta)j\Delta S \frac{V_{j+1}^i - V_{j-1}^i}{2\Delta S} - rV_j^i = 0 \]

the unknown here is $V_j^i$ as we have been working backward in time.
The discretised BSM equation is

$$\frac{V_{j}^{i+1} - V_{j}^{i}}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j}^{i+1} - 2V_{j}^{i+1} + V_{j}^{i+1}}{(\Delta S)^2}$$

$$+ (r - \delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_{j}^{i} = 0$$

Need to find $V_{j}^{i}$ so rearrange in terms of this unknown:

$$V_{j}^{i} = \frac{1}{1 + r\Delta t} (AV_{j+1}^{i+1} + BV_{j}^{i+1} + CV_{j-1}^{i+1}) \quad (*)$$

where:

$$A = (\frac{1}{2}\sigma^2 j^2 + \frac{1}{2}(r - \delta)j)\Delta t$$

$$B = 1 - \sigma^2 j^2 \Delta t$$

$$C = (\frac{1}{2}\sigma^2 j^2 - \frac{1}{2}(r - \delta)j)\Delta t$$
Thus is just like a binomial tree:

- we have a way of calculating the option value at expiry
- and we have scheme for calculating the option value at the previous time step.
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The differences between the binomial and explicit finite difference method are
- the binomial uses two nodes to the explicit finite difference’s three.
- You get to choose the specifications of the grid in the finite difference method
- You also need to specify the behaviour on the upper and lower $S$ boundaries.
The grid again:

Impose upper boundary at $S^U$

Use difference eq. (*) in “interior” of region, for $j = 1, \ldots, j_{\text{max}} - 1$

$$V^i_j = \frac{1}{1 + r\Delta t} \left( A V^{i+1}_{j+1} + B V^i_j + C V^{i+1}_{j-1} \right)$$

Impose lower boundary at 0
If we attempt to use equation (*) to calculate $V^i_0$ then we need to have values of $V^i_{-1}$ which we don’t have (e.g. for calls):

So for $V^i_0$ and $V^i_{j_{max}}$ we need to use our boundary conditions.

- $V^i_0 = 0$
- $V^i_{j_{max}} = S^u e^{-\delta(T-i\Delta t)} - X e^{-r(T-i\Delta t)}$

These conditions will naturally be different for different options, such as barrier options, put options etc.
The explicit finite difference scheme is like a trinomial tree.
Note that $A + B + C = 1$.
We can also show that the expected value of $S$ is at time $i\Delta t$:

$$E[S^i_j] = \frac{1}{1 + r\Delta t} E[S^{i+1}_j]$$

the expected future value of $S$, following GBM, under the risk-neutral probability discounted at the risk-free rate.
PROBABILISTIC INTERPRETATION

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  \[
  E[S^i_j] = \frac{1}{1 + r\Delta t} E[S^{i+1}_j]
  \] (1)
  the expected future value of $S$, following GBM, under the risk-neutral probability discounted at the risk-free rate.
- $A$, $B$ and $C$ can then be interpreted as the risk-neutral probabilities.
Unfortunately, the explicit method may be **unstable** – this means small errors **magnify** during the iterative procedure.
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Probabilistic ideas can be used to derive conditions for stability.

If we consider $A$, $B$ and $C$ as probabilities, we require that $A, B, C \geq 0$.

For $A$ and $C$ this requires:

$$j > \left| \frac{r - \delta}{\sigma^2} \right|$$
A far bigger problem is for $B$ where this says that

$$\Delta t < \frac{1}{\sigma^2 j^2}$$

which means that you need to ensure that the time interval is small enough.

The stability therefore restricts your choice of $\Delta t, \Delta S$
- $\Delta t$ cannot be too small, or else computation will take too long
- then this puts lower bound on size of $\Delta S$
How can we analyse the accuracy of the method?
CONVERGENCE

- How can we analyse the accuracy of the method?
- The errors will arise from only approximating the derivatives, in particular, in the explicit finite difference method:

\[
\frac{\partial^2 V}{\partial S^2}(S, t) = \frac{V(S + \Delta S, t) - 2V(S, t) + V(S - \Delta S, t)}{(\Delta S)^2} + O((\Delta S)^2)
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How can we analyse the accuracy of the method?

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Further analysis shows that the errors decrease linearly in time steps and quadratic in steps in $S$. 
NONLINEARITY ERROR

- Theoretical convergence rates depend upon all of the derivatives being well behaved (e.g. not infinite).
- However, we know that in the case of European options, the payoff at expiry is discontinuous leading to an infinite first derivative - and so it seems likely that our approximation may not work as well here.
Nonlinearity error

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- However, we know that in the case of European options, the payoff at expiry is discontinuous leading to an infinite first derivative - and so it seems likely that our approximation may not work as well here.
- There are therefore problems with any option that introduces a new boundary.
**SUMMARY**

- Introduced the finite-difference method to solve PDEs
- Discretise the original PDE to obtain a linear system of equations to solve.
- This scheme was explained for the Black Scholes PDE and in particular we derived the explicit finite difference scheme to solve the European call and put option problems.
SUMMARY

- Introduced the finite-difference method to solve PDEs
- Discretise the original PDE to obtain a linear system of equations to solve.
- This scheme was explained for the Black Scholes PDE and in particular we derived the explicit finite difference scheme to solve the European call and put option problems.
- The convergence of the method is similar to the binomial tree and, in fact, the method can be considered a trinomial tree.
- Explicit method can be unstable - constraints on our grid size.